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I.—ON THE MOTION OF A PARTICLE ALONG VARIABLE AND MOVEABLE TUBES AND SURFACES.

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THE earliest problem of the motion of a particle along a moveable tube of invariable form, is one given by John Bernoulli, (*Opera*, tom. 4, p. 248,) where the tube is rectilinear, and is made to revolve in a horizontal plane about one extremity with a uniform angular velocity. A solution of this problem is given also by Clairaut, to whom it had probably been proposed by Bernoulli, in the *Mémoires de l'Académie des Sciences de Paris*, for the year 1742, p. 10. A similar problem, which had been erroneously attempted by Barbier in the *Annales de Gergonne*, tom. 19, was correctly solved in the following volume by Ampère: in this problem the tube is supposed to revolve uniformly in a vertical instead of a horizontal plane, about the fixed extremity, the particle being consequently subject to the action of gravity. In the last number of this Journal may be seen a solution of this problem, by Professor Booth, who has discussed at length the more interesting cases of the motion. Several problems of a like character are to be met with, in which the tube is of invariable form, and is made to revolve about a fixed point with a uniform angular velocity. The object of this paper is to give a general method for the determination of the motion of a particle within tubes, and between contiguous surfaces of which either the position, or the form, or both, are made to

vary according to any assigned law whatever, the particle being acted on by any given forces. We will commence with the consideration of the motion of a particle along a tube, and for the sake of perfect generality, we will suppose the tube to be one of double curvature. The tube is considered in all cases to be indefinitely narrow and perfectly smooth, and every section at right angles to its axis to be circular.

Let the particle be referred to three fixed rectangular axes, and let x, y, z be its co-ordinates at any time t; let x, y, z become $x + \delta x$, $y + \delta y$, $z + \delta z$, when t becomes $t + \delta t$; δt, and consequently $\delta x, \delta y, \delta z$ being considered to be indefinitely small. Then the effective accelerating forces on the particle parallel to the three fixed axes will be at the time t,

$$\frac{\delta^2 x}{\delta t^2}, \quad \frac{\delta^2 y}{\delta t^2}, \quad \frac{\delta^2 z}{\delta t^2}.$$

Also, let X, Y, Z, represent the impressed accelerating forces on the particle resolved parallel to the axes of x, y, z; and let $x + dx$, $y + dy$, $z + dz$, be the co-ordinates of a point in the tube very near to the point x, y, z, which the particle occupies at the time t. Then, observing that the action of the tube on the particle is always at right angles to its axis at every point and therefore at the time t to the line joining the two points x, y, z, and $x + dx$, $y + dy$, $z + dz$, we have, by D'Alembert's Principle, combined with the Principle of Virtual Velocities,

$$\left(\frac{\delta^2 x}{\delta t^2} - X\right) dx + \left(\frac{\delta^2 y}{\delta t^2} - Y\right) dy + \left(\frac{\delta^2 z}{\delta t^2} - Z\right) dz = 0 \; \ldots . \; \text{(I.)}.$$

Again, since the form and position of the tube are supposed to vary according to some assigned law, it is clear that when t is known the equations to the tube must be known; hence it is evident that in addition to the equation (I.) we shall have, from the particular conditions of each individual problem, a number of equations equivalent to two of the form

$$\phi(x, y, z, t) = 0, \quad \chi(x, y, z, t) = 0 \; \ldots . \; \text{(II.)},$$

where ϕ and χ are symbols of functionality depending upon the law of the variations of the form and position of the tube.

The three equations (I.) and (II.) involve the four quantities x, y, z, t, and therefore in any particular case, if the difficulty of the analytical processes be not insuperable, we may ascertain x, y, z, each of them in terms of t; in which consists the complete solution of the problem.

If the tube remain during the whole of the motion within one plane; then the plane of x, y, being so chosen as to coincide with this plane, the three equations (I.) and (II.) will evidently be reduced to the two

$$\left(\frac{\delta^2 x}{\delta t^2} - X\right) dx + \left(\frac{\delta^2 y}{\delta t^2} - Y\right) dy = 0 \;\ldots\; \text{(III.)},$$

$$\phi(x, y, t) = 0 \;\ldots\ldots\ldots\; \text{(IV.)}$$

We proceed to illustrate the general formulæ of the motion by the discussion of a few problems.

1. A rectilinear tube revolves with a uniform angular velocity about one extremity in a horizontal plane; to find the motion of a particle within the tube. This is Bernoulli's problem.

Let ω be the constant angular velocity; r the distance of the particle at any time t from the fixed extremity of the tube; then the plane of x, y, being taken horizontal, and the origin of co-ordinates at the fixed extremity of the tube, we shall have, supposing the tube initially to coincide with the axis of x,

$$x = r\cos\omega t \;\ldots\; (1),$$

$$y = r\sin\omega t \;\ldots\; (2).$$

From (1) we have

$$dx = dr\cos\omega t,$$

and from (2),

$$dy = dr\sin\omega t.$$

Again, from (1) we have

$$\frac{\delta x}{\delta t} = \frac{\delta r}{\delta t}\cos\omega t - \omega r\sin\omega t,$$

$$\frac{\delta^2 x}{\delta t^2} = \frac{\delta^2 r}{\delta t^2}\cos\omega t - 2\omega\frac{\delta r}{\delta t}\sin\omega t - \omega^2 r\cos\omega t;$$

and from (2),

$$\frac{\delta y}{\delta t} = \frac{\delta r}{\delta t}\sin\omega t + \omega r\cos\omega t,$$

$$\frac{\delta^2 y}{\delta t^2} = \frac{\delta^2 r}{\delta t^2}\sin\omega t + 2\omega\frac{\delta r}{\delta t}\cos\omega t - \omega^2 r\sin\omega t.$$

Substituting in the general formula (III.) the values which we have obtained for dx, dy, $\frac{\delta^2 x}{\delta t^2}$, $\frac{\delta^2 y}{\delta t^2}$, we have, since $X = 0$, $Y = 0$,

$$\cos \omega t \left(\frac{\delta^2 r}{\delta t^2} \cos \omega t - 2\omega \frac{\delta r}{\delta t} \sin \omega t - \omega^2 r \cos \omega t \right)$$

$$+ \sin \omega t \left(\frac{\delta^2 r}{\delta t^2} \sin \omega t + 2\omega \frac{\delta r}{\delta t} \cos \omega t - \omega^2 r \sin \omega t \right) = 0;$$

and therefore

$$\frac{\delta^2 r}{\delta t^2} - \omega^2 r = 0:$$

the integral of this equation is

$$r = C\epsilon^{\omega t} + C'\epsilon^{-\omega t}.$$

Let $r = a$ when $t = 0$; then

$$a = C + C';$$

also let $\frac{\delta r}{\delta t} = \beta$ when $t = 0$; then

$$\beta = C\omega - C'\omega;$$

from the two equations for the determination of C and C', we have

$$C = \frac{a\omega + \beta}{2\omega}, \quad C' = \frac{a\omega - \beta}{2\omega};$$

hence for the motion of the particle along the tube

$$2\omega r = (a\omega + \beta)\,\epsilon^{\omega t} + (a\omega - \beta)\,\epsilon^{-\omega t}.$$

2. In the case of Ampère's problem, we shall have by the same process, the axis of y being now taken vertical, observing that $X = 0$, $Y = -g \sin \omega t$, if the time be reckoned from the moment of coincidence of the tube with the axis of x which is horizontal,

$$\frac{\delta^2 r}{\delta t^2} - \omega^2 r = -g \sin \omega t.$$

The integral of this equation is

$$r = C\epsilon^{\omega t} + C'\epsilon^{-\omega t} + \frac{g}{2\omega^2} \sin \omega t;$$

and if we determine the constants from the conditions that r, $\frac{dr}{dt}$ shall have initially values a, β, we shall get for the motion along the tube,

$$2\omega r = \left(a\omega + \beta - \frac{g}{2\omega} \right) \epsilon^{\omega t} + \left(a\omega - \beta + \frac{g}{2\omega} \right) \epsilon^{-\omega t} + \frac{g}{2\omega^2} \sin \omega t.$$

3. Suppose the tube to revolve in a horizontal plane about a fixed extremity with such an angular velocity, that the tangent of its angle of inclination to the axis of x is proportional to the time.

The equation to the tube at any time t will be

$$y = mtx \ldots\ldots (1),$$

where m is some constant quantity; hence

$$dy = mt\, dx,$$

and therefore from (III.), since $X = 0$ and $Y = 0$,

$$\frac{\delta^2 x}{\delta t^2} + mt \frac{\delta^2 y}{\delta t^2} = 0 \ldots\ldots (2).$$

But from (1) we have

$$\frac{\delta y}{\delta t} = mt \frac{\delta x}{\delta t} + mx,$$

$$\frac{\delta^2 y}{\delta t^2} = mt \frac{\delta^2 x}{\delta t^2} + 2m \frac{\delta x}{\delta t};$$

hence from (2),

$$(1 + m^2 t^2) \frac{\delta^2 x}{\delta t^2} + 2m^2 t \frac{\delta x}{\delta t} = 0,$$

$$\frac{\dfrac{\delta^2 x}{\delta t^2}}{\dfrac{\delta x}{\delta t}} + \frac{2m^2 t}{1 + m^2 t^2} = 0.$$

Integrating, we have

$$\log \frac{\delta x}{\delta t} + \log(1 + m^2 t^2) = \log C,$$

$$\frac{\delta x}{\delta t}(1 + m^2 t^2) = C.$$

Let β be the initial value of $\dfrac{\delta x}{\delta t}$, which will be the velocity of projection along the tube; then $C = \beta$, and therefore

$$\frac{\delta x}{\delta t}(1 + m^2 t^2) = \beta,\ \delta x = \frac{\beta \delta t}{1 + m^2 t^2};$$

integrating, we get

$$x + C = \frac{\beta}{m} \tan^{-1}(mt).$$

Let $x = a$ when $t = 0$; then $a + C = 0$, and therefore

$$x = a + \frac{\beta}{m} \tan^{-1}(mt),$$

and consequently from (1)

$$y = amt + \beta t \tan^{-1}(mt).$$

If θ be the inclination of the tube to the axis of x at any time, and r be the distance of the particle from the fixed extremity,

$$r = \frac{am + \beta\theta}{m \cos\theta}.$$

4. A circular tube is constrained to move in a horizontal plane with a uniform angular velocity about a fixed point in its circumference; to determine the motion of a particle within the tube, which is placed initially at the extremity of the diameter passing through the fixed point.

Let the fixed point be taken as the origin of co-ordinates, and let the axis of x coincide with the initial position of the diameter through this point; let ω be the angular velocity of the revolution of the circle, a the radius; also let θ be the angle at any time t between the distance of the particle and of the extremity of the diameter through the origin from the centre of the circle.

Then it will be easily seen that

$$x = a\cos\omega t + a\cos(\omega t - \theta) \ldots\ldots (1),$$

$$y = a\sin\omega t + a\sin(\omega t - \theta) \ldots\ldots (2).$$

From (1) we have

$$dx = a d\theta \sin(\omega t - \theta),$$

and from (2),

$$dy = -a d\theta \cos(\omega t - \theta).$$

Hence from (III.), observing that $X = 0$ and $Y = 0$,

$$\sin(\omega t - \theta)\frac{\delta^2 x}{\delta t^2} - \cos(\omega t - \theta)\frac{\delta^2 y}{\delta t^2} = 0 \ldots\ldots (3).$$

Again, from (1),

$$\frac{\delta x}{\delta t} = -a\omega\sin\omega t + a\left(\frac{\delta\theta}{\delta t} - \omega\right)\sin(\omega t - \theta),$$

$$\frac{\delta^2 x}{\delta t^2} = -a\omega^2\cos\omega t - a\left(\frac{\delta\theta}{\delta t} - \omega\right)^2\cos(\omega t - \theta) + a\frac{\delta^2\theta}{\delta t^2}\sin(\omega t - \theta);$$

and from (2),

$$\frac{\delta y}{\delta t} = a\omega\cos\omega t - a\left(\frac{\delta\theta}{\delta t} - \omega\right)\cos(\omega t - \theta),$$

$$\frac{\delta^2 y}{\delta t^2} = -a\omega^2\sin\omega t - a\left(\frac{\delta\theta}{\delta t} - \omega\right)^2\sin(\omega t - \theta) - a\frac{\delta^2\theta}{\delta t^2}\cos(\omega t - \theta);$$

and therefore by (3),

$$a\omega^2\{\sin\omega t\cos(\omega t-\theta)-\cos\omega t\sin(\omega t-\theta)\}+a\frac{\delta^2\theta}{\delta t^2}=0,$$

$$\omega^2\sin\theta+\frac{\delta^2\theta}{\delta t^2}=0:$$

multiplying by $2\dfrac{\delta\theta}{\delta t}$, and integrating,

$$\frac{\delta\theta^2}{\delta t^2}=C+2\omega^2\cos\theta.$$

But the absolute velocity of the particle being initially zero, it is clear that 2ω will be the initial value of $\dfrac{\delta\theta}{\delta t}$; and therefore, θ being initially zero, we have

$$4\omega^2=C+2\omega^2,\quad C=2\omega^2,$$

and therefore

$$\frac{\delta\theta^2}{\delta t^2}=2\omega^2(1+\cos\theta)=4\omega^2\cos^2\frac{\theta}{2},\quad \frac{\delta\theta}{\delta t}=2\omega\cos\frac{\theta}{2},$$

$$\frac{\cos\frac{\theta}{2}\,\delta\theta}{\cos^2\frac{\theta}{2}}=2\omega\delta t,\quad \frac{\delta\sin\frac{\theta}{2}}{1-\sin^2\frac{\theta}{2}}=\omega\delta t.$$

Integrating, we have

$$\log\frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}=2\omega t+C;$$

but $\theta=0$ when $t=0$; hence $C=0$, and we have

$$\frac{1+\sin\frac{\theta}{2}}{1-\sin\frac{\theta}{2}}=\varepsilon^{2\omega t},$$

and therefore

$$\sin\frac{\theta}{2}=\frac{\varepsilon^{\omega t}-\varepsilon^{-\omega t}}{\varepsilon^{\omega t}+\varepsilon^{-\omega t}},$$

which determines the position of the particle within the tube at any time. When $t=\infty$, we have $\sin\dfrac{\theta}{2}=1$, and therefore $\theta=\pi$, which shews that, after the lapse of an infinite time, the particle will arrive at the point of rotation.

5. If we pursue the same course as in the solution of the problems (1), (2), (4), we may obtain a convenient formula for the following more general problem: a plane curvilinear tube of any invariable form whatever revolves in its own plane about a fixed point with a uniform angular velocity; to determine the motion of a particle acted on by any forces within the tube.

Let ω be the constant angular velocity of the tube about the fixed point; r the distance of the particle at any time from this point; ϕ the angle between the simultaneous directions of r and of a line joining an assigned point of the tube with the fixed point of rotation; ds an element of the length of the tube at the place of the particle, and S the accelerating force on the particle resolved along the element ds; then the equation for the motion of the particle will be

$$r^2 \frac{\delta\phi^2}{\delta t^2} + \frac{\delta r^2}{\delta t^2} - \omega^2 r^2 = 2\int S \frac{ds}{d\phi}\, \delta\phi\,;$$

but since, the form of the tube being invariable, $\delta\phi$, δr may evidently be replaced by $d\phi$, dr, we have, putting, for the sake of uniformity of notation, dt in place of δt,

$$r^2 \frac{d\phi^2}{dt^2} + \frac{dr^2}{dt^2} - \omega^2 r^2 = 2\int S ds.$$

If ω be zero, the formula will become

$$r^2 \frac{d\phi^2}{dt^2} + \frac{dr^2}{dt^2} = 2\int S\, ds,$$

the well-known formula for the motion of a particle under the action of any forces within an immoveable plane tube.

6. In the foregoing examples the position of the tube varies with the time, the form however remains invariable. We will now give an example in which the form changes with the time.

A particle is projected with a given velocity within a circular tube, the radius of which increases in proportion to the time while the centre remains stationary; to determine the motion of the particle, the tube being supposed to lie always in a horizontal plane.

The equation to the circle will be

$$x^2 + y^2 = a^2 (1 + \alpha t)^2 \;\ldots\ldots\ldots\ldots\ldots\; (1),$$

where a and α are some constant quantities; hence

$$x\, dx + y\, dy = 0,$$

and therefore, by the general formula (III),

$$y\frac{\delta^2 x}{\delta t^2} - x\frac{\delta^2 y}{\delta t^2} = 0;$$

integrating, we have

$$y\frac{\delta x}{\delta t} - x\frac{\delta y}{\delta t} = C.$$

Let the axis of x be so chosen as to coincide with the initial distance of the particle from the centre, and let β be the initial velocity of the particle along the tube; then $C = -a\beta$, and therefore

$$x\frac{\delta y}{\delta t} - y\frac{\delta x}{\delta t} = a\beta \ldots\ldots\ldots\ldots (2);$$

again, from (1) we have

$$x\frac{\delta x}{\delta t} + y\frac{\delta y}{\delta t} = a^2\mathfrak{a}(1 + \mathfrak{a}t). \ldots (3);$$

multiplying (2) by y and (3) by x, and subtracting the former result from the latter, we have

$$(x^2 + y^2)\frac{\delta x}{\delta t} = a^2\mathfrak{a}(1 + \mathfrak{a}t)x - a\beta y,$$

and therefore by (1)

$$a(1 + \mathfrak{a}t)^2\frac{\delta x}{\delta t} = a\mathfrak{a}(1 + \mathfrak{a}t)x - \beta\left\{a^2(1 + \mathfrak{a}t)^2 - x^2\right\}^{\frac{1}{2}}.$$

Put $1 + \mathfrak{a}t = \tau$, then

$$a\mathfrak{a}\tau^2\frac{\delta x}{\delta\tau} = a\mathfrak{a}\tau x - \beta(a^2\tau^2 - x^2)^{\frac{1}{2}};$$

again, put $x = m\tau$, and there is

$$a\mathfrak{a}\tau^2\left(m + \tau\frac{\delta m}{\delta\tau}\right) = a\mathfrak{a}m\tau^2 - \beta\tau(a^2 - m^2)^{\frac{1}{2}},$$

$$a\mathfrak{a}\tau^3\frac{\delta m}{\delta\tau} = -\beta\tau(a^2 - m^2)^{\frac{1}{2}},$$

$$-a\mathfrak{a}\frac{\delta m}{(a^2 - m^2)^{\frac{1}{2}}} = \beta\frac{\delta\tau}{\tau^2};$$

integrating,

$$C + a\mathfrak{a}\cos^{-1}\frac{m}{a} = -\frac{\beta}{\tau},$$

or, putting for m its value,

$$C + aa \cos^{-1} \frac{x}{a\tau} = -\frac{\beta}{\tau},$$

and putting for τ its value $1 + at$,

$$C + aa \cos^{-1} \frac{x}{a(1+at)} = -\frac{\beta}{1+at}.$$

Now $x = a$ when $t = 0$; hence $C = -\beta$, and therefore

$$aa \cos^{-1} \frac{x}{a(1+at)} = \frac{a\beta t}{1+at},$$

$$x = a(1+at) \cos \frac{\beta t}{a(1+at)},$$

and therefore from (1)

$$y = a(1+at) \sin \frac{\beta t}{a(1+at)};$$

which give the absolute position of the particle at any assigned time.

We proceed now to the consideration of the motion of a particle along a surface from which it is unable to detach itself, while the surface itself changes its position or its form, or both, according to any assigned law. To fix the ideas, we suppose the particle to move between two surfaces indefinitely close together, so as to be expressed by the same equation.

Let x, y, z be the co-ordinates of the particle at any time t; and let $\delta x, \delta y, \delta z$ be the increments of x, y, z, in an indefinitely small time δt; also let dx, dy, dz denote the increments of x, y, z, in passing from the point x, y, z, to any point near to it within the surface as it exists at the time t. Also let X, Y, Z denote the resolved parts of the accelerating forces on the particle at the time t parallel to the axes of x, y, z; then, observing that the action of the surface on the particle is always in the direction of the normal at each point, we have, by D'Alembert's Principle combined with the Principle of Virtual Velocities,

$$\left(\frac{\delta^2 x}{\delta t^2} - X\right) dx + \left(\frac{\delta^2 y}{\delta t^2} - Y\right) dy + \left(\frac{\delta^2 z}{\delta t^2} - Z\right) dz = 0. \ldots (\text{A}).$$

Again, since the position and form of the surface vary according to an assigned law, its equation must evidently be known at any given time, and therefore we must have, from the nature of each particular problem, certain conditions between the quantities x, y, z, t, equivalent to a single equation

$$F = f(x, y, z, t) = 0 \ldots\ldots\ldots\ldots\ldots (\text{B}).$$

Taking the total differential of (B), we have

$$\frac{dF}{dx}dx + \frac{dF}{dy}dy + \frac{dF}{dz}dz = 0;$$

eliminating dz between this equation and (A), we get

$$\left(\frac{\delta^2 x}{\delta t^2} - X\right)\frac{dF}{dz}dx + \left(\frac{\delta^2 y}{\delta t^2} - Y\right)\frac{dF}{dz}dy$$
$$= \left(\frac{\delta^2 z}{\delta t^2} - Z\right)\left(\frac{dF}{dx}dx + \frac{dF}{dy}dy\right);$$

but dx and dy are independent quantities, we have therefore, by equating separately their coefficients on each side of the equation,

$$\left(\frac{\delta^2 x}{\delta t^2} - X\right)\frac{dF}{dz} = \left(\frac{\delta^2 z}{\delta t^2} - Z\right)\frac{dF}{dx},$$

$$\left(\frac{\delta^2 y}{\delta t^2} - Y\right)\frac{dF}{dz} = \left(\frac{\delta^2 z}{\delta t^2} - Z\right)\frac{dF}{dy},$$

and therefore also

$$\left(\frac{\delta^2 x}{\delta t^2} - X\right)\frac{dF}{dy} = \left(\frac{\delta^2 y}{\delta t^2} - Y\right)\frac{dF}{dx}:$$

any two of these three relations, together with the equation (B), will give us three equations in x, y, z, t, whence x, y, z are to be determined in terms of t.

The following example will serve to illustrate the use of these equations. We have taken a case where the form of the surface remains invariable, its position alone being liable to change. The analysis however, in the solution of problems of the class which we are considering, receives its general character solely in consequence of the presence of t in the equation (B), and therefore the example which we have chosen is sufficient for the general object we have in view.

A particle descends by the action of gravity down a plane which revolves uniformly about a vertical axis through which it passes; to determine the motion of the particle.

Let the plane of x, y be taken horizontal, the axis of x coinciding with the initial intersection of the revolving plane with the horizontal plane through the origin, and let the axis of z be taken vertically downwards; then, ω denoting the angular velocity of the plane, its equation at any time t will be

$$F = y\cos\omega t - x\sin\omega t = 0 \; \ldots\ldots\ldots\ldots \; (1),$$

whence

$$\frac{dF}{dx} = -\sin\omega t, \quad \frac{dF}{dy} = \cos\omega t, \quad \frac{dF}{dz} = 0;$$

also $X = 0$, $Y = 0$, $Z = g$; and therefore, from either of the two first of the three general relations,

$$\frac{\delta^2 z}{\delta t^2} = g \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and from the third,

$$\frac{\delta^2 x}{\delta t^2} \cos \omega t + \frac{\delta^2 y}{\delta t^2} \sin \omega t = 0 \ldots\ldots\ldots\ldots (3).$$

Let r denote the distance of the particle at any time from the axis of z; then

$$x = r \cos \omega t, \; y = r \sin \omega t,$$

whence

$$\frac{\delta x}{\delta t} = \frac{\delta r}{\delta t} \cos \omega t - \omega r \sin \omega t,$$

$$\frac{\delta^2 x}{\delta t^2} = \frac{\delta^2 r}{\delta t^2} \cos \omega t - 2\omega \frac{\delta r}{\delta t} \sin \omega t - \omega^2 r \cos \omega t,$$

$$\frac{\delta y}{\delta t} = \frac{\delta r}{\delta t} \sin \omega t + \omega r \cos \omega t,$$

$$\frac{\delta^2 y}{\delta t^2} = \frac{\delta^2 r}{\delta t^2} \sin \omega t + 2\omega \frac{\delta r}{\delta t} \cos \omega t - \omega^2 r \sin \omega t;$$

and therefore from (3)

$$\frac{\delta^2 r}{\delta t^2} - \omega^2 r = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Let the initial values of z, $\frac{dz}{dt}$ be 0, β; and those of r, $\frac{\delta r}{\delta t}$ be a, α; then, from the equations (2) and (4), after executing obvious operations, we shall obtain

$$z = \tfrac{1}{2} g t^2 + \beta t,$$

$$2\omega r = (\omega a + \alpha)\,\varepsilon^{\omega t} + (\omega a - \alpha)\,\varepsilon^{-\omega t},$$

and

$$\log \frac{(\omega^2 r^2 + \alpha^2 - \omega^2 a^2)^{\frac{1}{2}} + \omega r}{\alpha + \omega a} = \frac{\omega}{g} \left\{(2gz + \beta^2)^{\frac{1}{2}} - \beta\right\};$$

the two first of these equations give the position of the particle on the revolving plane, and therefore, by virtue of the equation (1), the absolute position of the particle at any time; while the third is the equation to the path which the particle describes on the plane.

II.—REMARKS ON THE BINOMIAL THEOREM.*

THE proof given by Euler of the Binomial Theorem appears to depend upon a casual result of multiplication, and has sometimes been called *tentative*. This proof may be seen, in an extended form, in M. Lebefure de Fourcy's treatise on Algebra; but the apparent *casualty* still remains. In the following sketch, this defect, for such we may suppose it to be from the remark it has excited, is removed to the extent of placing the general theorem on the same footing as its particular case when the exponent is an integer; and so as to embrace the more extended form just alluded to.

1. In the common multiplication of $a+b$ by itself repeatedly, the process shows that the successive coefficients are as at the side, where each one is made by adding the one vertically above it to the one above it on the left.

1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1

But if m_n signify the number of ways in which m can be selected out of n, it is obvious that

$$m_n = m_{n-1} + (m-1)_{n-1};$$

whence it easily follows that the m^{th} *variable* number in the n^{th} row is m_n. Hence the binomial theorem easily follows for an integer exponent.

Now let there be a succession of symbols, A, B, C, &c. such that the change which converts A into B, converts B into C, C into D, and so on: and let a, b, c, &c. be another set having the same property. Taking the expression $A+a$, make a succession of similar operations as follows:—each operation consists in making the change with respect to A, and multiplying by A, doing the same with a, and adding the results. Let Θ be the symbol of this operation. We have then

$$\begin{aligned}\Theta(A+a) &= AB + Aa \\ &\quad + Aa + ab = AB + 2Aa + ab, \\ \Theta^2(A+a) &= ABC + 2ABa + Aab \\ &\quad + ABa + 2Aab + abc \\ &= ABC + 3ABa + 3Aab + abc,\end{aligned}$$

and so on. When $ABC\ldots$ and $abc\ldots$ contain n factors, let them be A_n and a_n; we have then

$$\Theta^n(A+a) = A_n + nA_{n-1}a + n\frac{n-1}{2}A_{n-2}a_2 + \ldots + a_n,$$

which is an extension of the binomial theorem. Observe that

* From a Correspondent.

the right of $\Theta^n (A+a)$ to the designation $(A+a)_n$ is not universal.

Now let

$B = A + \delta$, $C = A + 2\delta$, &c., and $b = a + \delta$, $c = a + 2\delta$, &c.

we have

$\Theta(A+a) = (A+a)(A+a+\delta)$, &c. or $\Theta^n(A+a) = (A+a)_n$,

whence the above theorem is true when P_n means $P^{n|\delta}$, and there readily follows, $1.2.3.\ldots n$ being denoted by $[n]$,

$$\frac{(A+a)_n}{[n]} = \frac{A_n}{[n]} + \frac{A_{n-1}}{[n-1]}\frac{a_1}{[1]} + \frac{A_{n-2}}{[n-2]}\frac{a_2}{[2]} + \ldots + \frac{a_n}{[n]};$$

or the series $1 + (A+a)_1 x + (A+a)_2 \dfrac{x^2}{2} + \ldots$ is the product of the series $1 + A_1 x + A_2 \dfrac{x^2}{2} + \ldots$ and $1 + a_1 x + a_2 \dfrac{x^2}{2} + \ldots$;

from whence, by the usual consideration of the equation $\phi(A+a) = \phi A \times \phi a$, it follows that

$$\left\{1 + Ax + A(A+\delta)\frac{x^2}{2} + \ldots\right\}^m = 1 + mAx + mA(mA+\delta)\frac{x^2}{2} + \ldots$$

for all possible values of m. And $A = 1$ $\delta = -1$ gives the binomial theorem, while $Ax = 1$, $\delta = 0$ gives the exponential theorem.

A. D. M.

III.—ON THE PROPERTIES OF A CERTAIN SYMBOLICAL EXPRESSION.

By Arthur Cayley, B.A. Trinity College.

The series

$$S_{p\,0}^{\,\infty}\cdot \zeta_p (a^2 + b^2 \ldots n \text{ terms})^{p+i}\left(\frac{l}{1+l}\cdot\frac{d^2}{da^2} + \frac{m}{1+m}\cdot\frac{d^2}{db^2}\cdot\cdot\right)^p \times$$

$$\frac{1}{\{(1+l)a^2 + (1+m)b^2 \ldots\}^i}$$

$$\left(\zeta_p = \frac{1}{2^{2p+1}\,1.2\ldots p.\,i(i+1)\ldots(i+p)}\right)\ldots\ldots(\psi),$$

possesses some remarkable properties, which it is the object of the present paper to investigate. We shall prove that the symbolical expression (ψ) is independent of a, b, &c., and equivalent to the definite integral

$$\int_0^1 \frac{x^{2i-1}\,dx}{\{(1+lx^2)(1+mx^2)\ldots\}^{\frac{1}{2}}},$$

a property which we shall afterwards apply to the investigation of the attractions of an ellipsoid upon an external point, and to some other analogous integrals. The demonstration of this, which is one of considerable complexity, may be effected as follows:

Writing the symbol $\frac{l}{1+l} \cdot \frac{d^2}{da^2} + \frac{m}{1+m} \cdot \frac{d^2}{db^2} \dots$ under the form

$$\left(\frac{d^2}{da^2} + \frac{d^2}{db^2} + \frac{d^2}{dc^2}\right) - \left(\frac{1}{1+l} \cdot \frac{d^2}{da^2} + \frac{1}{1+m} \cdot \frac{d^2}{db^2} \dots\right)$$

$$= \Delta - \left(\frac{1}{1+l} \cdot \frac{d^2}{da^2} + \frac{1}{1+m} \cdot \frac{d^2}{db^2} \dots\right) \text{ suppose.}$$

Let the p^{th} power of this quantity be expanded in powers of Δ. The general term is

$$(-1)^q \cdot \frac{p \cdot (p-1) \dots (p-q+1)}{1 \cdot 2 \dots q} \cdot \Delta^{p-q} \cdot \left(\frac{1}{1+l} \cdot \frac{d^2}{da^2} \dots\right)^q,$$

which is to be applied to $\frac{1}{\{(1+l)\,a^2 \dots\}^i}$.

Considering the expression

$$\left(\frac{1}{1+l} \cdot \frac{d^2}{da^2} \dots\right)^q \cdot \frac{1}{\{(1+l)\,a^2 \dots\}^i};$$

if for a moment we write $(1+l)\,a^2 = a_1^2$, &c. $\Delta_1 = \frac{d^2}{da_1^2} + \frac{d^2}{db_1^2} \dots$ $\rho_1 = a_1^2 + b_1^2 + c_1^2 \dots$, this becomes

$$\Delta_1^q \cdot \frac{1}{\rho_1^i}.$$

Now it is immediately seen that $\Delta_1 \frac{1}{\rho_1^{i'}} = \frac{2i' \cdot (2i' + 2 - \mathfrak{n})}{\rho_1^{i'+1}}$;

from which we may deduce

$$\Delta_1^q \cdot \frac{1}{\rho_1^i} = \frac{2i.(2i+2)\dots(2i+2q-2)\,(2i+2-\mathfrak{n})\dots(2i+2q-\mathfrak{n})}{\rho_1^{i+q}},$$

or, restoring the value of ρ_1, and forming the expression for the general term of (ψ), this is

$$\zeta_p \cdot \rho^{\mathfrak{n}+1} \begin{cases} \Delta^p \cdot \dfrac{1}{(a^2 + b^2 \dots + la^2 + mb^2 + \&c.)^i} \\ -\dfrac{p}{1}\, 2i \cdot (2i + 2 - \mathfrak{n})\, \Delta^{p-1} \cdot \dfrac{1}{(a^2 + b^2 + \dots + la^2 + mb^2)^i} \\ + \&c. \end{cases}$$

ρ representing the quantity $a^2 + b^2 + \&c.$

Hence, selecting the terms of the s^{th} order in l, m, &c. the expression for the part of (ψ) which is of the s^{th} order in l, m, &c. may be written under the form

$$S_p\,{}_0^s\frac{(-1)^s\rho^{p+i}\zeta_p}{1\,.\,2\,.\,.\,.\,.\,s}$$

multiplied by

$$\left\{\begin{array}{l} i\,.\,(i+1)\,.\,.\,.\,.\,(i+s-1)\,\Delta^p\,.\,\dfrac{U}{\rho^{i+s}} \\ -\dfrac{p}{1}\,2i\,.\,(2i+2-\mathfrak{n})\,(i+1)\,.\,.\,.\,.\,(i+s)\,\Delta^{p-1}\,\dfrac{U}{\rho^{i+s+1}} \\ +\dfrac{p\,.\,p-1}{1\,.\,2}\,2i.(2i+2)\,(2i+2-\mathfrak{n})\,(2i+4-\mathfrak{n})\,(i+2).\,.(i+s+1)\,\Delta^{p-2}\dfrac{U}{\rho^{i+s+2}} \\ -\,\&\text{c.} \qquad\qquad [la^2+mb^2\,.\,.\,.\,.=U \text{ suppose}] \end{array}\right.$$

which for conciseness we shall represent by

$$\frac{(-1)^s}{1\,.\,2\,.\,.\,.\,.\,s}\,S_p\,{}_0^s\,\rho^{p+i}\,.\,\zeta_p\left\{\begin{array}{l} \alpha_s\,\Delta^p\,.\,\dfrac{U}{\rho^{i+s}} \\ -\dfrac{p}{1}\beta_s\,\Delta^{p-1}\,.\,\dfrac{U}{\rho^{i+s+1}} \\ +\dfrac{p\,.\,p-1}{1\,.\,2}\,\gamma_s\,.\,\Delta^{p-2}\,.\,\dfrac{U}{\rho^{i+s+2}} \\ -\,\&\text{c.} \end{array}\right.$$

$= S$ suppose.

Now U representing any homogeneous function of the order $2s$, it is easily seen that

$$\Delta\,\frac{U}{\rho^i}=\frac{\Delta U}{\rho^i}+2i\,.\,(2i+2-4s-\mathfrak{n})\,\frac{U}{\rho^{i+1}}\,.$$

And repeating continually the operation Δ, observing that ΔU, $\Delta^2 U$, &c. are of the orders $2(s-1)$, $2(s-2)$, &c. we at length arrive at

$$\begin{aligned} \Delta^q\,.\,\frac{U}{\rho^i} = {} & \Delta^q\,.\,U\,.\,\frac{1}{\rho^i} \\ & +\frac{q}{1}\,2i.(2i+2q-4s-\mathfrak{n})\,\Delta^{q-1}U.\frac{1}{\rho^{i+1}} \\ & +\frac{q.q-1}{1.2}\,2i.(2i+2)(2i+2q-4s-\mathfrak{n})(2i+2q-4s-\mathfrak{n}-2)\,\Delta^{q-2}.U.\frac{1}{\rho^{i+2}} \\ & \quad\vdots \\ & +2i\,.\,(2i+2).\,.(2i+2q)\,(2i+2q-4s-\mathfrak{n}).\,.(2i+2-4s-\mathfrak{n})\frac{U}{\rho^{i+q}}\,. \end{aligned}$$

Changing i into $s+i+i'$, we have an equation which we may represent by

$$\Delta^q \cdot \frac{U}{\rho^{s+i+i'}} = A_{q,i'} \frac{\Delta^q \cdot U}{\rho^{s+i+i'}} + {}^1A_{q,i'} \frac{\Delta^{q-1} \cdot U}{\rho^{s+i+i'+1}} \cdot\cdot + {}^qA_{q,i'} \frac{U}{\rho^{s+i+i'+q}} \cdot\cdot (a),$$

where in general

$$ {}^rA_{q,i'} = \frac{q \cdot (q-1) \ldots\ldots (q-r+1)}{1.2 \ldots\ldots r}$$

$$\times (2s+2i+2i')(2s+2i'+2) \ldots\ldots (2s+2i'+2i+2r-2)$$

$$\times (2i+2i'+2q-2s-\mathfrak{n}) \ldots\ldots (2i+2i'+2q-2s-\mathfrak{n}-2r+2).$$

Now the value of S, written at full length, is

$$\frac{(-1)^s}{1.2 \ldots\ldots s} \left\{ \begin{array}{l} \zeta_s \rho^{s+i} \left(a_s \Delta^s \frac{U}{\rho^{s+i}} + \frac{s}{1} \beta_s \Delta^{s-1} \frac{U}{\rho^{s+i+1}} \ldots\ldots \right. \\ + \zeta_{s-1} \rho^{s+i-1} \left(a_s \Delta^{s-1} \frac{U}{\rho^{s+i}} + \frac{s-1}{1} \beta_s \Delta^{s-2} \frac{U}{\rho^{s+i+1}} + \ldots\ldots \right. \\ + \&c. \end{array} \right.$$

and substituting for the several terms of this expansion the values given by the equation (a), we have

$$S = \frac{(-1)^s}{1.2 \ldots\ldots s} \left(k_0 \Delta^s U + k_1 \cdot \frac{1}{\rho} \Delta^{s-1} U \ldots\ldots + k_s \frac{1}{\rho^s} \cdot U \right)$$

where in general

$$k_{\mathfrak{x}} = a_s ({}^{\mathfrak{x}}A_{s,0} \zeta_s + {}^{\mathfrak{x}-1}A_{s-1,0} \zeta_{s-1} \ldots\ldots + A_{s-\mathfrak{x},0} \zeta_{s-\mathfrak{x}})$$

$$+ \beta_s \left(\frac{s}{1} {}^{\mathfrak{x}-1}A_{s-1,1} \zeta_s \ldots\ldots + A_{s-\mathfrak{x},1} \frac{(s-\mathfrak{x}+1)}{1} \zeta_{s-\mathfrak{x}+1} \right)$$

$$\vdots$$

$$+ \lambda_s \left(\frac{s \cdot (s-1) \ldots\ldots (s-\mathfrak{x}+1)}{1.2 \ldots\ldots \mathfrak{x}} A_{s-\mathfrak{x},\mathfrak{x}} \right),$$

λ_s being the $(\mathfrak{x}+1)^{\text{th}}$ of the series $a_s, \beta_s \ldots\ldots$

Substituting for the quantities involved in this expression, and putting, for simplicity, $2i+2-\mathfrak{n}=2\gamma$, we have, without any further reduction, except that of arranging the factors of the different terms, and cancelling those which appear in the numerator and denominator of the same term,

$$\frac{(-1)^s k_{\mathfrak{x}}}{1.2 \ldots\ldots s} =$$

$$\frac{(-1)^{s-\mathfrak{x}} (1-\gamma)(2-\gamma) \ldots\ldots (\lambda-\gamma)}{2^{2s+1} \cdot 1.2 \ldots\ldots s \cdot 1.2 \ldots\ldots (s-\mathfrak{x}) \cdot 1.2 \ldots\ldots \mathfrak{x}}$$

multiplied by the series

$$(i+s+1)\ldots(i+s+x-1)\left\{1+\frac{\gamma x}{1.(x-\gamma)}+\frac{1}{1.2}\,\frac{\gamma.(\gamma+1)\,x.(x-1)}{(x-\gamma)(x-1-\gamma)}+\ldots(x+1)\text{ terms}\right.$$

$$-\frac{(i+s)\ldots(i+s+x-2)}{1-\gamma}\left\{x+\frac{\gamma x.(x-1)}{1.(x-\gamma)}+\frac{1}{1.2}\,\frac{\gamma.(\gamma+1)\,.\,x.(x-1)(x-2)}{(x-\gamma)(x-1-\gamma)}\right.$$
$$+\ldots x\text{ terms}$$

$$+(-1)^r\frac{(i+s-r+1)\ldots(i+s+x-r-1)}{(1-\gamma)(2-\gamma)\ldots(r-\gamma)}\left\{x.(x-1)\ldots(x-r+1)+\frac{\gamma}{1}\,\frac{x.(x-1)\ldots(x-r)}{(x-\gamma)}\right.$$
$$\ldots x+1-r\text{ terms}$$

to $r=x$.

Now it may be shown that

$$\frac{1}{(1-\gamma)(2-\gamma)\ldots.(r-\gamma)}$$
$$\left\{x.(x-1)\,.\,.(x-r+1)+\frac{\gamma}{1}\,\frac{x.(x-1)\,.\,.\,.(x-r)}{x-\gamma}+\&c.\,.\,.(x+1-r)\text{ terms}\right\}$$
$$=\frac{x.(x-1)\ldots.(r+1)\,.\,x.(x-1)\ldots.(x-r+1)}{(1-\gamma)(2-\gamma)\ldots.(x-\gamma)},$$

which reduces the expression for k_x to the form

$$\frac{(-1)^s.k_x}{1.2.\,.s}=\frac{(-1)^{s+x}}{2^{2s+1}.\,1.2.\,.s.1.2.\,.(s-x)}\left\{\begin{array}{l}(i+s+1).\,.(i+s+x-1)\\ -(i+s).\,.(i+s+x-2)\dfrac{x}{1}\\ +(i+s-1).\,.(i+s+x-3)\dfrac{x.(x-1)}{1.2}\\ \&c.\ (x+1)\text{ terms};\end{array}\right.$$

from which it may be shown, that except for $x=0$, $k_x=0$.

The value $x=0$, observing that the expression

$$(i+s+1)(i+s+2)\ldots.(i+s-1)$$

represents $\dfrac{1}{i+s}$, gives

$$\frac{(-1)^s k_0}{1.2\ldots.s}=\frac{(-1)^s}{2^{2s}(1.2\ \ldots.\ s)^2.(2i+2s)};$$

or we have simply

$$S=\frac{(-1)^s}{2^{2s}(1.2\ldots.s)^2.(2i+2s)}\,\Delta^s.U,$$

where $\Delta = \frac{d^2}{da^2} + \frac{d^2}{db^2} + \ldots,\quad U = (la^2 + mb^2 \ldots)^s.$

Consider the term $\frac{1.2\ldots s}{1.2\ldots\lambda\,.\,1.2\ldots\mu\,.\,\&c.}\, a^{2\lambda}\,.\,b^{2\mu}\ldots l^{\lambda}\,.\,m^{\mu}\ldots$

With respect to this, Δ^s reduces itself to

$$\frac{1.2\ldots s}{1.2\ldots\lambda\,.\,1.2\ldots\mu\,.\,\&c.}\left(\frac{d}{da}\right)^{2\lambda}\ldots$$

and the corresponding term of S is

$$\frac{(-1)^s}{2^{2s}.(2i+2s)(1.2\ldots\lambda\,(1.2\ldots\mu\,\&c.)^2}\,1.2\ldots2\lambda\,.\,1.2\ldots2\mu\,\&c.\ l^{\lambda}.m^{\mu}\ldots$$

$$= \frac{(-1)^s.\,1.3\ldots(2\lambda-1)\,.\,1.3\ldots(2\mu-1)\,\&c.}{(2i+2s)\,2.4\ldots2\lambda\,.\,2.4\ldots2\mu\,\&c.}\,l^{\lambda}.m^{\mu}\ldots$$

which, omitting the factor $\frac{1}{2i+2s}$, and multiplying by x^{2s}, is the general term of the s^{th} order in l, m, n, of

$$\frac{1}{\sqrt{\{(1+lx^2)(1+mx^2)\ldots\}}}.$$

The term itself is therefore the general term of

$$\int_0^1 \frac{x^{2i-1}\,dx}{\sqrt{\{(1+lx^2)(1+mx^2)\ldots}};$$

or taking the sum of all such terms for the complete value of S, and the sum of the different values of S for s variable, we have the required equation

$$\psi = \int \frac{x^{2i-1}\,.\,dx}{\sqrt{\{(1+lx^2)(1+mx^2)\ldots}}.$$

Another and perhaps more remarkable form of this equation may be deduced by writing $\frac{a^2}{1+l}$, $\frac{b^2}{1+m}$, &c. for a^2, b^2, &c., and putting $\frac{a^2}{1+l} + \frac{b^2}{1+m} + \&c. = \eta^2,\ l\eta^2 = \alpha^2,\ m\eta^2 = \beta^2$, &c., we readily deduce

$$\eta^{n-2i}.\int_0^1 \frac{x^{2i-1}\,.\,dx}{\{(\eta^2+\alpha^2x^2)(\eta^2+\beta^2x^2)\ldots\}^{\frac{1}{2}}}$$

$$= S_p{}_0^\infty \frac{1}{2^{2p+1}\,.\,1.2\ldots p.i.(i+1)\ldots(i+p)}\left(\alpha^2\frac{d^2}{da^2}+\beta^2.\frac{d^2}{db^2}\ldots\right)^p \frac{1}{(a^2+b^2+c^2)^i},$$

η being determined by the equation

$$\frac{a^2}{\eta^2+\alpha^2} + \frac{b^2}{\eta^2+\beta^2}\ldots = 1;$$

or, as it may otherwise be written,

$$\eta^2 = a^2 + b^2 + c^2 - \frac{a^2\alpha^2}{\eta^2 + \alpha^2} - \frac{b^2\beta^2}{\eta^2+\beta^2} - \&c.$$

$\mathfrak{n}$, it will be recollected, denotes the number of the quantities a, b, &c.

Now suppose

$$V = \iint \ldots . \phi(a-x,\ b-y \ldots .)\, dx\, dy \ldots .$$

(the integral sign being repeated $\mathfrak{n}$ times) where the limits of the integral are given by the equation

$$\frac{x^2}{h^2} + \frac{y^2}{h_{,}^2} + \&c. = 1;$$

and that it is permitted, throughout the integral to expand the function $\phi(a-x, \ldots .)$ in ascending powers of x, y, &c. (the condition for which is apparently that of ϕ not becoming infinite for any values of x, y, &c., included within the limits of the integration): then observing that any integral of the form $\iint \ldots . x^p y^q \ldots dx\, dy$ &c. . where either p, q, &c. . . is odd, when taken between the required limits contains equal positive and negative elements, and therefore vanishes, the general term of V assumes the form

$$\frac{1}{1.2..2r.1.2..2s..}\left(\frac{d}{da}\right)^{2r}\left(\frac{d}{db}\right)^{2s}..\phi(a,b..)\iint..x^{2r}y^{2s}..dx\,dy..$$

Also, by a formula quoted in the eleventh number of the *Mathematical Journal*, the value of the definite integral $\iint..x^{2r}y^{2s}..dx\,dy..$is

$$h^{2r+1}.h_{,}^{2s+1}\ldots .\frac{\Gamma(r+\frac{1}{2}).\Gamma(s+\frac{1}{2})\ldots .}{\Gamma(r+s+\ldots .+\frac{\mathfrak{n}}{2}+1)},$$

(observing that the value there given referring to positive values only of the variables, must be multiplied by $2^{\mathfrak{n}}$): or, as it may be written,

$$h^{2r+1}.h_{,}^{2s+1}\ldots .\pi^{\frac{1}{2}\mathfrak{n}}.\frac{1}{2^{r+s}}\ldots \frac{1.3\ldots .(2r-1)\,1.3\ldots .(2s-1)..}{\frac{\mathfrak{n}}{2}.\left(\frac{\mathfrak{n}}{2}+1\right)\ldots\left(\frac{\mathfrak{n}}{2}+r+s\right)\ldots .\Gamma\left(\frac{\mathfrak{n}}{2}\right)},$$

whence the general term of V takes the form

$$\frac{hh_{,}..\pi^{\frac{1}{2}\mathfrak{n}}}{\Gamma\left(\frac{\mathfrak{n}}{2}\right)}.\frac{1}{\frac{\mathfrak{n}}{2}.\left(\frac{\mathfrak{n}}{2}+1\right)..\left(\frac{\mathfrak{n}}{2}+r+s\right)..}.\frac{1}{2^{2r+2s..}}\,\frac{1}{1.2.3..r.1.2..s..}$$

$$\times\left(h^2.\frac{d^2}{da^2}\right)^r.\left(h_{,}^2\frac{d^2}{db^2}\right)^s\ldots .\phi(a,b,\ldots .).$$

And putting $r + s + \&c. = p$, and taking the sum of the terms that answer to the same value of p, it is immediately seen that this sum is

$$= \frac{hh_{,}\ldots\pi^{\frac{1}{2}n}}{\Gamma\left(\frac{n}{2}\right)} \cdot \frac{1}{2^{2p}\cdot 1.2\ldots p\cdot\frac{n}{2}\cdot\left(\frac{n}{2}+1\right)\ldots\left(\frac{n}{2}+p\right)}$$

$$\left(h^2\frac{d^2}{da^2} + h_{,}^2\cdot\frac{d^2}{db^2}\cdot\ldots\right)^p\cdot\phi(a,b\,.\,.).$$

Or the function $\phi(a-x,\ b-y\ldots)$ not becoming infinite within the limits of the integration, we have

$$\int\int\ldots\phi(a-x,\ b-y\ldots)\,dx\,dy\ldots$$

$$= \frac{2hh_{,}\ldots\pi^{\frac{1}{2}n}}{\Gamma\left(\frac{n}{2}\right)} S_{p\,0}^{\,\infty}\frac{1}{2^{2p+1}.1.2\ldots p.\frac{n}{2}.\left(\frac{n}{2}+1\right).\left(\frac{n}{2}+p\right)}\left(h^2\frac{d^2}{da^2}\ldots\right)^p\phi(a,b\,.\,.).$$

The integral on the first side of the equation extending to all real values of x, y, &c., subject to $\frac{x^2}{h^2}+\frac{y^2}{h_{,}^2}+\ldots<1$. Suppose in the first place $\phi(a,b\ldots)=\frac{1}{(a^2+b^2\ldots)^{\frac{1}{2}n}}$.

By a preceding formula the second side of the equation reduces itself to

$$\frac{2hh_{,}\ldots\pi^{\frac{1}{2}n}}{\Gamma\left(\frac{n}{2}\right)}\cdot\int_0^1\frac{x^{n-1}\,.\,dx}{\sqrt{\{(\eta^2+h^2x^2)(\eta^2+h_{,}^2x^2)\ldots(n\text{ factors})\}}},$$

η being given by $\frac{a^2}{\eta^2+h^2}+\frac{b^2}{\eta^2+h_{,}^2}\ldots=1$.

Hence the formula

$$\iint\ldots,\ n\text{ times}\ \frac{dx\,dy\ldots}{\{(a-x)^2+(b-y)^2\ldots\}^{\frac{1}{2}n}}$$

$$= \frac{2hh_{,}\ldots\pi^{\frac{1}{2}n}}{\Gamma\left(\frac{n}{2}\right)}\int_0^1\frac{x^{n-1}\,.\,dx}{\sqrt{\{(\eta^2+h^2x^2)(\eta^2+h_{,}^2x^2)\ldots(n\text{ factors})\}}}.$$

The integral on the first side of the equation extending to all real values of x, y, &c. satisfying $\frac{x^2}{h^2}+\frac{y^2}{h_{,}^2}+\&c.\ldots<1$; η^2 we have seen being determined by

$$\frac{a^2}{\eta^2+h^2}+\frac{b^2}{\eta^2+h_{,}^2}+\&c.=1.$$

Finally, the condition of $\phi(a-x,\ b-y\ldots.)$ not becoming infinite within the limits of the integration, reduces itself to $\frac{a^2}{h^2}+\frac{b^2}{h_{,}^2}+\ldots.>1$, which must be satisfied by these quantities.

Suppose in the next place the function $\phi(a, b.\ldots)$ satisfies $\frac{d^2\phi}{da^2}+\frac{d^2\phi}{db^2}+\&c.=0$. The factor $(h^2\frac{d^2}{da^2}+\&c.)$ may be written under the form

$$(h_{,}^2-h^2)\frac{d^2}{db^2}+(h_{,,}^2-h^2)\frac{d^2}{dc^2}+\&c.+h^2\left(\frac{d^2}{da^2}+\frac{d^2}{db^2}\ldots.\right)$$

$$=(h_{,}^2-h^2)\frac{d^2}{db^2}+\&c.$$

Since, as applied to the function ϕ, $\frac{d^2}{da^2}+\frac{d^2}{db^2}+\&c.$ is equivalent to 0, we have in this case

$$\iint\ldots.\ \phi(a-x,\ b-y\ \ldots.)\,dx\,dy\ \ldots.$$

$$=\frac{2hh_{,}\ldots.\pi^{\frac{1}{2}n}}{\Gamma\left(\frac{n}{2}\right)}\cdot S_{p\,0}^{\ \infty}\frac{1}{2^{2p+1}\cdot 1.2\ \ldots.\ p\,\frac{n}{2}\cdot\cdot\left(\frac{n}{2}+p\right)}$$

$$\left\{(h_{,}^2-h^2)\frac{d^2}{db^2}+\ldots.\right\}^p\cdot\phi(a, b\ \ldots.);$$

or the first side divided by $hh_{,}\ldots.$ has the remarkable property of depending on the differences $h_{,}^2-h^2$, &c. only the generalisation of a well known property of the function V, in the theory of the attraction of a spheroid upon an external point. If in this equation we put $\phi(a,b.\ldots)=\frac{a}{(a^2+b^2\ \ldots)^n}$, which satisfies the required condition $\frac{d^2\phi}{da^2}+\&c.=0$, we have transferring (a) to the left hand side of the sign S, and putting in a preceding formula, $a^2=0$, $\beta^2=h_{,}^2-h^2$, &c. and η^2+h^2 for η^2,

$$\iint\ldots.\ (n\text{ times})\cdot\frac{(a-x)\,dx\,dy\ldots.}{\{(a-x)^2+(b-y)^2\ \ldots.\}^{\frac{1}{2}n}}$$

$$=\frac{2hh_{,}\ldots.\pi^{\frac{1}{2}n}a}{\sqrt{(\eta^2+h^2)}\cdot\Gamma\left(\frac{n}{2}\right)}$$

$$\int_0^1\frac{x^{n-1}\cdot dx}{[\{\eta^2+h^2+(h_{,}^2-h^2)x^2\}.\{\eta^2+h^2+(h_{,,}^2-h^2)x^2\}\ \ldots(n-1)\text{ factors}]^{\frac{1}{2}}},$$

where, as before, the integrations on the first side extend to all real values of x, y, &c., satisfying $\frac{x^2}{h^2} + \frac{y^2}{h_i^2} \ldots . < 1$; η^2 is determined by $\frac{a^2}{\eta^2 + h^2} +$ &c. $= 1$. And $a, b \ldots . h, h_1$ &c. are subject to $\frac{a^2}{h^2} + \frac{b^2}{h_i^2} +$ &c. $\ldots . > 1$.

For $n = 3$, this becomes,

$$\iiint \frac{(a-x)\,dx\,dy\,dz}{\{(a-x)^2 + (b-y)^2 + (c-z)^2\}^{\frac{3}{2}}} =$$

$$\frac{4\pi h h_i h_{ii}}{\sqrt{(h^2+\eta^2)}} \cdot \int_0^1 \frac{x^2 dx}{\sqrt{[\{\eta^2 + h^2 + (h_i^2 - h^2)x^2\}\{\eta^2 + h^2 + (h_{ii}^2 - h^2)x^2\}]}}:$$

the integrations on the first side extending over the ellipsoid whose semiaxes are h, h_i, h_{ii}, and the point whose coordinates are a, b, c, being exterior to this ellipsoid. Also $\frac{a^2}{\eta^2 + h^2} + \frac{b^2}{\eta^2 + h_i^2} + \frac{c^2}{\eta^2 + h_{ii}^2} = 1$: a known theorem.

IV.—ON THE UNIFORM MOTION OF HEAT IN HOMOGENEOUS SOLID BODIES, AND ITS CONNECTION WITH THE MATHEMATICAL THEORY OF ELECTRICITY.*

[Since the following article was written, the writer finds that most of his ideas have been anticipated by M. Chasles in two Mémoires in the *Journal de Mathématiques;* the first in Vol. III., on the Determination of the Value of a certain Definite Integral, and the second, in Vol. V., on a new Method of Determining the attraction of an Ellipsoid on a Point without it. In the latter of these Mémoires, M. Chasles refers to a paper, by himself, in the twenty-fifth *Cahier* of the *Journal de l'Ecole Polytechnique,* in which it is probable there are still farther anticipations, though the writer of the present article has not had access to so late a volume of the latter journal. Since, however, most of his methods are very different from those of M. Chasles, which are nearly entirely geometrical, the following article may be not uninteresting to some readers.]

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* From a Correspondent.

If an infinite homogeneous solid be submitted to the action of certain constant sources of heat, the stationary temperature at any point will vary according to its position; and, through every point there will be a surface, over the whole extent of which, the temperature is constant, which is therefore called an *isothermal* surface. In this paper the case will be considered, in which these surfaces are finite, and consequently closed.

It is obvious that the temperature of any point without a given isothermal surface, depends merely on the form and temperature of the surface, being independent of the actual sources of heat by which this temperature is produced, provided there are no sources without the surface. The temperature of an external point is consequently the same as if all the sources were distributed over this surface, in such a manner as to produce the given constant temperature. Hence we may consider the temperature of any point without the isothermal surface, as the sum of the temperatures due to certain constant sources of heat, distributed over that surface.

To find the temperature produced by a single source of heat, let r be the distance of any point from it, and let v be the temperature at that point. Then, since the temperature is the same for all points situated at the same distance from the source, it is readily shown, that v is determined by the equation

$$-r^2 \frac{dv}{dr} = A.$$

Dividing both members by r^2, and integrating, we have

$$v = \frac{A}{r} + C.$$

Now, let us suppose, that the natural temperature of the solid, or the temperature at an infinite distance from the source, is zero: then we shall have $C = 0$, and consequently

$$v = \frac{A}{r} \; \ldots\ldots \; (1).$$

Hence, that part of the temperature of a point without an isothermal surface which is due to the sources of heat situated on any element, $d\omega_1^2$, of the surface, is $\frac{\rho_1 d\omega_1^2}{r_1}$, where r_1 is the distance from the element to that point, and ρ_1 a quantity measuring the intensity of the sources of heat, at different parts of the surface. Hence, the supposition being still made,

that there are no sources of heat without the surface, if v be the temperature at the external point, we have

$$v = \iint \frac{\rho_1 d\omega_1{}^2}{r_1} \;\ldots\ldots\; (2),$$

the integrals being extended over the whole surface. The quantity ρ_1 must be determined by the condition

$$v = v_1 \;\ldots\ldots (3),$$

for any point in the surface, v_1 being a given constant temperature.

Let us now consider what will be the temperature of a point within the surface, supposing all the sources of heat by which the surface is retained at the temperature v_1, to be distributed over it. Since there are no sources in the interior of the surface, it follows, that as much heat must flow out from the interior across the surface, as flows into the interior, from the sources of heat at the surface. Hence the total flux of heat from the original surface, to an adjacent isothermal surface, in the interior, is nothing. Hence also the flux of heat from this latter surface, to an adjacent isothermal surface, in its interior, must be nothing; and so on through the whole of the body within the original surface. Hence the temperature in the interior is constant, and equal to v_1, and therefore, for points at the surface, or within it, we have

$$\iint \frac{\rho_1 d\omega_1{}^2}{r_1} = v_1 \;\ldots\ldots\; (4).$$

Now, if we suppose the surface to be covered with an attractive medium, whose density at different points is proportional to ρ_1, $-\frac{d}{dx}\iint \rho_1 \frac{d\omega_1{}^2}{r_1}$ will be the attraction, in the direction of the axis of x, on a point whose rectangular co-ordinates are x, y, z. Hence it follows, that the attraction of this medium on a point within the surface is nothing, and consequently ρ_1 is proportional to the intensity of electricity, in a state of equilibrium on the surface, the attraction of electricity in a state of equilibrium being nothing on an interior point. Since, at the surface, the value of $\iint \frac{\rho_1 d\omega_1{}^2}{r_1}$ is constant, and since, on that account, its value within the surface is constant also, it follows, that if the attractive force on a point at the surface is perpendicular to the surface, the attraction on a point within the surface is nothing. Hence the sole condition of

equilibrium of electricity, distributed over the surface of a body, is, that it must be so distributed that the attraction on a point at the surface, oppositely electrified, may be perpendicular to the surface.

Since, at any of the isothermal surfaces, v is constant, it follows, that $-\frac{dv}{dn}$, where n is the length of a curve which cuts all the surfaces perpendicularly, measured from a fixed point to the point attracted, is the total attraction on the latter point; and that this attraction is in a tangent to the curve n, or in a normal to the isothermal surface passing through the point. For the same reason also, if ρ_1 represent a flux of heat, and not an electrical intensity, $-\frac{dv}{dn}$ will be the total flux of heat at the variable extremity of n, and the direction of this flux will be along n, or perpendicular to the isothermal surface. Hence, if a surface in an infinite solid be retained at a constant temperature, and if a conducting body, bounded by a similar surface, be electrified, the flux of heat, at any point, in the first case, will be proportional to the attraction on an electrical point, similarly situated, in the second; and the direction of the flux will correspond to that of the attraction.

Let $-\frac{dv_1}{dn_1}$ be the external value of $-\frac{dv}{dn}$, at the original surface, or the attraction on a point without it, and indefinitely near it. Now this attraction is composed of two parts; one the attraction of the adjacent element of the surface; and the other the attraction of all the rest of the surface. Hence, calling the former of these a, and the latter b, we have

$$-\frac{dv_1}{dn_1} = a + b.$$

Now, since the adjacent element of the surface may be taken as infinitely larger, in its linear dimensions, than the distance from it of the point attracted, its attraction will be the same as that of an infinite plane, of the density ρ_1. Hence a is independent of the distance of the point from the surface, and is equal to $2\pi\rho_1$. Hence

$$-\frac{dv_1}{dn_1} = 2\pi\rho_1 + b.$$

Now, for a point within the surface, the attraction of the adjacent element will be the same, but in a contrary direction, and the attraction of the rest of the surface will be the same, and in the same direction. Hence the attraction on a point within

the surface, and indefinitely near it, is $-2\pi\rho_1 + b$; and consequently, since this is equal to nothing, we must have $b = 2\pi\rho_1$, and therefore

$$-\frac{dv_1}{dn_1} = 4\pi\rho_1 \ldots . (5).$$

Hence ρ_1 is equal to the total flux of heat, at any point of the surface, divided by 4π.

It also follows, that if the attraction of matter spread over the surface be nothing on an interior point, the attraction on an exterior point, indefinitely near the surface, is perpendicular to the surface, and equal to the density of the matter at the part of the surface adjacent to that point, multiplied by 4π.

If v be the temperature at any isothermal surface, and ρ the intensity of the sources at any point of this surface, which would be necessary to sustain the temperature v, we have, by (5),

$$-\frac{dv}{dn} = 4\pi\rho,$$

which equation holds, whatever be the manner in which the actual sources of heat are arranged, whether over an isothermal surface, or not; and the temperature produced, in an external point, by the former sources, is the same as that produced by the latter. Also, the total flux of heat across the isothermal surface, whose temperature is v, is equal to the total flux of heat from the actual sources. From this, and from what has been proved above, it follows, that if a surface be described round a conducting or non-conducting electrified body, so that the attraction on points situated on this surface may be every where perpendicular to it, and if the electricity be removed from the original body, and distributed in equilibrium over this surface, its intensity at any point will be equal to the attraction of the original body on that point, divided by 4π, and its attraction on any point without it will be equal to the attraction of the original body on the same point.

If we call E the total expenditure of heat, or the whole flux across any isothermal surface, we have, obviously,

$$E = -\iint \frac{dv_1}{dn_1}\, d\omega^2{}_1.$$

Now this quantity should be equal to the sum of the expenditures of heat from all the sources. To verify this, we must, in the first place, find the expenditure of a single source. Now the temperature produced by a single source is, by (1),

$v = \dfrac{A}{r}$, and hence the expenditure is obviously equal to $-\dfrac{dv}{dr} \times 4\pi r^2$, or to $4\pi A$. If $A = \rho_1\, d\omega_1^{\,2}$, this becomes $4\pi\rho_1 d\omega_1^{\,2}$. Hence the total expenditure is $\iint 4\pi\rho_1\, d\omega_1^{\,2}$, or $-\iint \dfrac{dv_1}{dn_1}\, d\omega_1^{\,2}$, which agrees with the expression found above.

The following is an example of the application of these principles.

Uniform Motion of Heat in an Ellipsoid.

The principles established above, afford an easy method of determining the isothermal surfaces, and the corresponding temperatures, in the case in which the original isothermal surface is an ellipsoid.

The first step is to find ρ_1, which is proportional to the quantity of matter at any point in the surface of an ellipsoid, when the matter is so distributed, that the attraction on a point within the ellipsoid is nothing. Now the attraction of a shell, bounded by two concentric similar ellipsoids, on a point within it, is nothing. If the shell be infinitely thin, its attraction will be the same as that of matter distributed over the surface of one of the ellipsoids, in such a manner, that the quantity at any point is proportional to the thickness of the shell at the same point. Let a_1, b_1, c_1, be the semiaxes of one of the ellipsoids, $a_1 + \delta a_1$, $b_1 + \delta b_1$, $c_1 + \delta c_1$, those of the other. Let also p_1 be the perpendicular from the centre to the tangent plane, at any point on the first ellipsoid, and $p_1 + \delta p_1$ the perpendicular from the centre to the tangent plane, at a point similarly situated on the second. Then δp_1 is the thickness of the shell, since, the two ellipsoids being similar, the tangent planes at the points similarly situated on their surfaces, are parallel. Also, on account of their similarity, $\dfrac{\delta a_1}{a_1} = \dfrac{\delta b_1}{b_1} = \dfrac{\delta c_1}{c_1} = \dfrac{\delta p_1}{p_1}$, and consequently the thickness of the shell is proportional to p_1. Hence we have, by (5),

$$-\frac{1}{4\pi}\frac{dv_1}{dn_1} = \rho_1 = k_1 p_1 \; \ldots\ldots \; (a),$$

where k_1 is a constant, to be determined by the condition $v = v_1$, at the surface of the ellipsoid.

To find the equation of the isothermal surface at which the temperature is $v_1 + dv_1$, let $-dv_1 = C$, in (a). Then we have

$k_1\, p_1\, dn_1 = \dfrac{C}{4\pi}$, or $p_1\, dn_1 = \theta_1$, where θ_1 is an infinitely small constant quantity; and the required equation will be the equation of the surface traced by the extremity of the line dn_1, drawn externally perpendicular to the ellipsoid. Let x', y', z', be the co-ordinates of any point in that surface, and x, y, z, those of the corresponding points in the ellipsoid. Then, calling $\alpha_1, \beta_1, \gamma_1$, the angles which a normal to the ellipsoid at the point whose co-ordinates are x, y, z, makes with these co-ordinates, and supposing the axes of x, y, z, to coincide with the axes of the ellipsoid, $2a_1$, $2b_1$, $2c_1$, respectively, we have

$$x' - x = dn_1 \cos\alpha_1 = \frac{\dfrac{x}{a_1^2}\, dn_1}{\sqrt{\left(\dfrac{x^2}{a_1^4} + \dfrac{y^2}{b_1^4} + \dfrac{z^2}{c_1^4}\right)}} = \frac{x}{a_1^2}\, p_1\, dn_1 = \frac{x}{a_1^2}\, \theta_1.$$

or $x' - x = \dfrac{x'}{a_1^2}\, \theta_1$, since θ_1 is infinitely small, and therefore also $x' - x$; whence

$$x = x'\left(1 - \frac{\theta_1}{a_1^2}\right) = \frac{x'}{1 + \dfrac{\theta_1}{a_1^2}}.$$

In a similar manner we should find

$$y = \frac{y'}{1 + \dfrac{\theta_1}{b_1^2}}, \qquad z = \frac{z'}{1 + \dfrac{\theta_1}{c_1^2}}.$$

But $\dfrac{x^2}{a_1^2} + \dfrac{y^2}{b_1^2} + \dfrac{z^2}{c_1^2} = 1$, and hence we have

$$\frac{x'^2}{a_1^2\left(1 + \dfrac{\theta_1}{a_1^2}\right)^2} + \frac{y'^2}{b_1^2\left(1 + \dfrac{\theta_1}{b_1^2}\right)^2} + \frac{z'^2}{c_1^2\left(1 + \dfrac{\theta_1}{c_1^2}\right)^2} = 1,$$

$$\text{or } \frac{x'^2}{a_1^2 + 2\theta_1} + \frac{y'^2}{b_1^2 + 2\theta_1} + \frac{z'^2}{c_1^2 + 2\theta_1} = 1,$$

for the equation to the isothermal surface whose temperature is $v_1 + dv_1$, and which is therefore an ellipsoid described from the same foci as the original isothermal ellipsoid. In exactly the same manner it might be shown, that the isothermal surface whose temperature is $v_1 + dv_1 + dv_1'$, is an ellipsoid having the same foci as the ellipsoid whose temperature is $v_1 + dv_1$, and, consequently, as the original ellipsoid also. By continu-

ing this process it may be proved, that all the isothermal surfaces are ellipsoids, having the same foci as the original one.

From the form of the equation found above for the isothermal ellipsoid whose temperature is $v_1 + dv_1$, it follows, that θ_1 or $p_1\, dn_1$ is $= a_1\, da_1$, where da_1 is the increment of a_1, corresponding to the increment dn_1, of n_1. Hence, if a be one of the semiaxes of an ellipsoid, $a + da$ the corresponding semiaxis of another ellipsoid, having the same foci, dn the thickness at any point of the shell bounded by the two ellipsoids, and p the perpendicular from the centre to the plane touching either ellipsoid at the same point, we have

$$\frac{dn}{da} = \frac{a}{p} \;\ldots\ldots\; (b).$$

All that remains to be done is to find the temperature at the surface of any given ellipsoid, having the same foci as the original ellipsoid. For this purpose, let us first find the value of $-\dfrac{dv}{dn}$ at any point in the surface of the isothermal ellipsoid whose semiaxes are a, b, c. Now, we have, from (a),

$$-\frac{dv}{dn} = 4\pi kp,$$

where k is constant for any point in the surface of the isothermal ellipsoid under consideration, and determined by the condition, that the whole flux of heat across this surface must be equal to the whole flux across the surface of the original ellipsoid. Now the first of these quantities is equal to $4\pi k \iint p\, d\omega^2$, ($d\omega^2$ being an element of the surface) or to $4\pi\dfrac{ka}{\delta a}\iint \delta p\, d\omega^2$, since $\dfrac{\delta a}{a} = \dfrac{\delta p}{p}$. But $\iint \delta p\, d\omega^2$ is equal to the volume of a shell bounded by two similar ellipsoids, whose semiaxes are a, b, c, and $a + \delta a$, $b + \delta b$, $c + \delta c$, and is therefore readily shown to be equal to $4\pi\dfrac{\delta a}{a}abc$. Hence $4\pi\dfrac{ka}{\delta a}\iint \delta p\, d\omega^2$, or $4\pi k \iint p\, d\omega^2$ is equal to $4^2\pi^2 k\, abc$. In a similar manner we have, for the flux of heat across the original isothermal surface, $4^2\pi^2 k_1\, a_1 b_1 c_1$, and therefore

$$4^2\pi^2 kabc = 4^2\pi^2 k_1 a_1 b_1 c_1,$$

$$\text{which gives } k = k_1\frac{a_1 b_1 c_1}{abc}.$$

Hence, we have

$$-\frac{dv}{dn} = 4\pi k_1 \frac{a_1 b_1 c_1}{abc} p \ldots\ldots (c).$$

The value of v may be found by integrating this equation. To effect this, since a, b, c are the semiaxes of an ellipsoid passing through the variable extremity of n, and having the same foci as the original ellipsoid, whose axes are a_1, b_1, c_1, we have

$$a^2 - a_1^2 = b^2 - b_1^2 = c^2 - c_1^2;$$

$$\left.\begin{array}{r} \text{which gives} \quad b^2 = a^2 - f^2 \\ c^2 = a^2 - g^2 \\ \text{where} \quad f^2 = a_1^2 - b_1^2, \quad g^2 = a_1^2 - c_1^2 \end{array}\right\} \ldots\ldots (d).$$

Hence (c) becomes

$$-\frac{dv}{dn} = 4\pi k_1 \frac{a_1 b_1 c_1 p}{a\sqrt{(a^2 - f^2)}\sqrt{(a^2 - g^2)}}.$$

Now, by (b), $dn = \dfrac{ada}{p}$, and hence

$$dv = -4\pi k_1 \frac{a_1 b_1 c_1 \, da}{\sqrt{(a^2 - f^2)}\sqrt{(a^2 - g^2)}}.$$

Integrating this, we have

$$v = -4\pi k_1 a_1 b_1 c_1 \int \frac{da}{\sqrt{(a^2 - f^2)}\sqrt{(a^2 - g^2)}} + C \ldots . (e).$$

The two constants, k_1 and C, must be determined by the conditions $v = v_1$ when $a = a_1$, and $v = 0$ when $a = \infty$; the latter of which must be fulfilled, in order that the expression found for v may be equal to $\displaystyle\iint \frac{k_1 p_1 d\omega_1^2}{r_1}$.

To reduce the expression for v to an elliptic function, let us assume

$$\left.\begin{array}{l} a = f \operatorname{cosec} \phi \\ a_1 = f \operatorname{cosec} \phi_1 \end{array}\right\} \ldots\ldots\ldots (f),$$

which we may do with propriety, if f be the greater of the two quantities f and g, since a is always greater than either of them, as we see from (d). On this assumption, equation (e) becomes

$$v = \frac{4\pi k_1 a_1 b_1 c_1}{f} \int_0^\phi \frac{d\phi}{\sqrt{(1 - c'^2 \sin^2 \phi)}} + C = \frac{4\pi k_1 a_1 b_1 c_1}{f} F_{c'}\phi + C$$

$$\text{where} \quad c' = \frac{g}{f} \ldots\ldots\ldots\ldots (g).$$

Determining from this the values of C and k_1, by the conditions mentioned above, we find $C = 0$, and

$$k_1 = \frac{fv_1}{4\pi a_1 b_1 c_1 F_{c'}\phi_1} \ldots\ldots\ldots (h);$$

hence, the expression for v becomes

$$v = v_1 \frac{F_{c'}\phi}{F_{c'}\phi_1} \ldots\ldots\ldots\ldots (k).$$

The results which have been obtained may be stated as follows:—

If, in an infinite solid, the surface of an ellipsoid be retained at a constant temperature, the temperature of any point in the solid will be the same as that of any other point in the surface of an ellipsoid described from the same foci, and passing through that point; and the flux of heat at any point in the surface of this ellipsoid will be proportional to the perpendicular from the centre to a plane touching it at the point, and inversely proportional to the volume of the ellipsoid.

This case of the uniform motion of heat was first solved by Lamé, in his Mémoire on Isothermal Surfaces, in Liouville's *Journal de Mathématiques*, Vol. II., p. 147, by showing, that a series of isothermal surfaces of the second order will satisfy the equation

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = 0,$$

provided they are all described from the same foci. The value which he finds for v agrees with (e), and he finds, for the flux of heat at any point, the expression

$$\frac{KA}{\sqrt{(\mu^2 - \nu^2)}\,\sqrt{(\mu^2 - \rho^2)}};$$

or, according to the notation which we have employed,

$$\frac{4\pi k_1 a_1 b_1 c_1}{\sqrt{(a^2-\nu^2)}\,\sqrt{(a^2-\rho^2)}},$$

where ν is the greater real semiaxis of the hyperboloid of one sheet, and ρ the real semiaxis of the hyperboloid of two sheets, described from the same foci as the original ellipsoid, and passing through the point considered. Hence a^2, ν^2, ρ^2 are the three roots of the equation

$$\frac{x^2}{u} + \frac{y^2}{u - f^2} + \frac{z^2}{u - g^2} = 1,$$

or $u^3 - (f^2+g^2+x^2+y^2+z^2)\,u^2 + \{f^2g^2 + (f^2+g^2)x^2 + g^2y^2 + f^2z^2\}\,u - f^2g^2x^2 = 0.$

Hence $a^2\nu^2\rho^2 = f^2g^2x^2$,

and $a^2\nu^2 + a^2\rho^2 + \nu^2\rho^2 = f^2g^2 + (f^2+g^2)\,x^2 + g^2y^2 + f^2z^2$.

Therefore,

$$(a^2-\nu^2)(a^2-\rho^2) = a^4 - a^2\nu^2 - a^2\rho^2 - \nu^2\rho^2 + \frac{2a^2\nu^2\rho^2}{a^2}$$

$$= a^4 - \{f^2g^2 + (f^2+g^2)\,x^2 + g^2y^2 + f^2z^2\} + 2\,f^2g^2\,\frac{x^2}{a^2}$$

$$= a^4 - (a^2-b^2)(a^2-c^2) - (2a^2-b^2-c^2)\,x^2 - (a^2-c^2)\,y^2$$
$$- (a^2-b^2)\,z^2 + 2\,(a^2-b^2)(a^2-c^2)\frac{x^2}{a^2}$$

$$= a^4 - (a^2-b^2)(a^2-c^2) - (b^2+c^2)\,x^2 - (a^2-c^2)\,y^2$$
$$- (a^2-b^2)\,z^2 + 2\,b^2c^2\left(1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)$$

$$= a^4 - (a^2-b^2)(a^2-c^2) - (b^2+c^2)\,x^2 - (a^2+c^2)\,y^2 - (a^2+b^2)\,z^2 + 2\,b^2c^2$$
$$= a^2b^2 + a^2c^2 + b^2c^2 - \{(b^2+c^2)\,x^2 + (a^2+c^2)\,y^2 + (a^2+b^2)\,z^2\};$$

which is readily shown, by substituting for $a^2b^2 + a^2c^2 + b^2c^2$ its equal $(a^2b^2 + a^2c^2 + b^2c^2)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$, to be equal to $\frac{a^2b^2c^2}{p^2}$.

Hence the expression for $-\frac{dv}{dn}$, given above, becomes

$$-\frac{dv}{dn} = 4\pi k_1\,\frac{a_1b_1c_1}{abc}\,p,$$

which agrees with (c).

Attraction of a Homogeneous Ellipsoid on a Point within or without it.

If, in (c), we put $k_1 = \frac{da_1}{a_1}$, the value of $-\frac{dv}{dn}$ at any point will be the attraction on that point, of a shell bounded by two similar concentric ellipsoids, whose semiaxes are

$$a_1,\ a_1\sqrt{(1-e^2)},\ a_1\sqrt{(1-e'^2)},$$

and $\quad a_1 + da_1,\ (a_1+da_1)\sqrt{(1-e^2)},\ (a_1+da_1)\sqrt{(1-e'^2)},$

$$\left.\begin{aligned} \text{where }\ a^2 - b^2 &= a_1^2 - b_1^2 = a_1^2e^2 \\ \text{and }\ a^2 - c^2 &= a_1^2 - c_1^2 = a_1^2e'^2 \end{aligned}\right\} \ldots\ldots (1),$$

the density of the shell being unity. Now this attraction is in a normal drawn through the point attracted, to the surface of the ellipsoid whose semiaxes are a, b, c. If we call

α, β, γ, the angles which this normal makes with the co-ordinates x, y, z, of the point attracted, we have

$$\cos \alpha = \frac{\dfrac{x}{a^2}}{\sqrt{\left(\dfrac{x^2}{a^4} + \dfrac{y^2}{b^4} + \dfrac{z^2}{c^4}\right)}} = \frac{px}{a^2},$$

$$\text{and similarly,} \quad \cos \beta = \frac{py}{b^2}, \ \cos \gamma = \frac{pz}{c^2}.$$

Hence, calling dA, dB, dC, the components of the attraction parallel to the axes of co-ordinates, we have, from (c),

$$\left.\begin{aligned} dA &= 4\pi x \frac{b_1 c_1}{a^3 bc} p^2 da_1 \\ dB &= 4\pi y \frac{b_1 c_1}{ab^3 c} p^2 da_1 \\ dC &= 4\pi z \frac{b_1 c_1}{abc^3} p^2 da_1 \end{aligned}\right\} \ldots\ldots (2).$$

The integrals of these expressions, between the limits $a_1 = 0$, and $a_1 = a_1'$, are the components of the attraction of an ellipsoid whose semiaxes are a_1', b_1', c_1', or a_1', $a_1'\sqrt{(1-e^2)}$, $a_1'\sqrt{(1-e'^2)}$, on the point (x, y, z). Now, by (1), we may express each of the quantities b, c, b_1, c_1, in terms of a and a_1, and the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \ \text{ or } \ \frac{x^2}{a^2} + \frac{y^2}{a^2 - e^2 a_1^2} + \frac{z^2}{a^2 - e'^2 a_1^2} = 1 \ldots . (3),$$

enables us to express either of the quantities a, a_1, in terms of the other. The simplest way, however, to integrate equations (2), will be to express each in terms of a third quantity,

$$u = \frac{a_1}{a} \ldots\ldots\ldots (4).$$

Eliminating a from (3), by means of this quantity, we have

$$a_1^2 = u^2 x^2 + \frac{y^2}{u^{-2} - e^2} + \frac{z^2}{u^{-2} - e'^2}.$$

$$\text{Hence } \ a_1 da_1 = \left\{ux^2 + \frac{u^{-3}y^2}{(u^{-2} - e^2)^2} + \frac{u^{-3}z^2}{(u^{-2} - e'^2)^2}\right\} du$$

$$= \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right) a_1^4 u^{-3} du = a_1^4 p^{-2} u^{-3} du.$$

Also, from (4), we have $a = \frac{a_1}{u}$; from which we find, by (1), $b = \frac{a_1}{u}\sqrt{(1 - e^2u^2)}$, $c = \frac{a_1}{u}\sqrt{(1 - e'^2u^2)}$. By (1) also, $b_1 = a_1\sqrt{(1 - e^2)}$, $c_1 = a_1\sqrt{(1 - e'^2)}$. Making these substitutions in (2), and integrating, we have, calling a' the value of a, when $a_1 = a_1'$,

$$\left.\begin{aligned} A &= 4\pi x \sqrt{(1-e^2)}\sqrt{(1-e'^2)}\int_0^{\frac{a_1'}{a'}} \frac{u^2du}{\sqrt{(1-e^2u^2)}\sqrt{(1-e'^2u^2)}} \\ B &= 4\pi y \sqrt{(1-e^2)}\sqrt{(1-e'^2)}\int_0^{\frac{a_1'}{a'}} \frac{u^2du}{(1-e^2u^2)^{\frac{3}{2}}(1-e'^2u^2)^{\frac{1}{2}}} \\ C &= 4\pi z \sqrt{(1-e^2)}\sqrt{(1-e'^2)}\int_0^{\frac{a_1'}{a'}} \frac{u^2du}{(1-e^2u^2)^{\frac{1}{2}}(1-e'^2u^2)^{\frac{3}{2}}} \end{aligned}\right\} \ldots\ldots(5).$$

If the point attracted be within the ellipsoid, the attraction of all the similar concentric shells without the point will be nothing; and hence the superior limit of u will be the value of $\frac{a_1}{a}$ at the surface of an ellipsoid, similar to the given one, and passing through the point attracted.

Now, in this case, $a_1 = a$, since a is one of the semiaxes of an ellipsoid passing through the point attracted, and having the same foci as another ellipsoid (passing through the same point), whose corresponding semiaxis is a_1. Hence, for an interior point, we have

$$\left.\begin{aligned} A &= 4\pi x \sqrt{(1-e^2)}\sqrt{(1-e'^2)}\int_0^{1} \frac{u^2du}{\sqrt{(1-e^2u^2)}\sqrt{(1-e'^2u^2)}} \\ B &= 4\pi y \sqrt{(1-e^2)}\sqrt{(1-e'^2)}\int_0^{1} \frac{u^2du}{(1-e^2u^2)^{\frac{3}{2}}(1-e'^2u^2)^{\frac{1}{2}}} \\ C &= 4\pi z \sqrt{(1-e^2)}\sqrt{(1-e'^2)}\int_0^{1} \frac{u^2du}{(1-e^2u^2)^{\frac{1}{2}}(1-e'^2u^2)^{\frac{3}{2}}} \end{aligned}\right\} \ldots\ldots(6).$$

These are the known expressions for the attraction of an ellipsoid on a point within it. Equations (5) agree with the expressions given in the Supplement to Liv. v. of Pontécoulant's "*Théorie Analytique du Système du Monde,*" where they are found by direct integration, by a method discovered by Poisson. They may also be readily deduced from equations (6), by Ivory's Theorem. Or, on the other hand, by a

comparison of them, after reducing the limits of the integrals to 0 and 1, by substituting $\dfrac{a'_1}{a'}v$ for u, with equation (6), Ivory's Theorem may be readily demonstrated.

P. Q. R.

V.—ON THE LIMITS OF MACLAURIN'S THEOREM.

By A. Q. G. Craufurd, M.A. of Jesus College.

To express by a single term the series which remains after the first n terms of Maclaurin's series are taken.

Let $\overset{a}{\underset{n}{C}}u$ denote the coefficient of a^n in the development of a function of a which is represented by u.

Let $f(x)$ represent any function of x which is developable in a series of positive ascending powers of x; and, first, suppose the series to be finite, and to contain $m+1$ terms.

Then,

$$f(x) = \overset{a}{\underset{0}{C}}f(a) + x\overset{a}{\underset{1}{C}}f(a) + x^2\overset{a}{\underset{2}{C}}f(a)\ \ldots\ldots + x^n\overset{a}{\underset{n}{C}}f(a)$$

$$+ x^{n+1}\overset{a}{\underset{n+1}{C}}f(a) + x^{n+2}\overset{a}{\underset{n+2}{C}}f(a)\ \ldots\ldots\ x^m\overset{a}{\underset{m}{C}}f(a).$$

Now $\overset{a}{\underset{n+1}{C}}f(a) = \overset{a}{\underset{n}{C}}\dfrac{f(a)}{a}$, and $\overset{a}{\underset{n+p}{C}}f(a) = \overset{a}{\underset{n}{C}}\dfrac{f(a)}{a^p}$.

Therefore the second line of the series is equivalent to

$$x^{n+1}\left\{\overset{a}{\underset{n+1}{C}}f(a) + x\overset{a}{\underset{n+1}{C}}\frac{f(a)}{a} + x^2\overset{a}{\underset{n+1}{C}}\frac{f(a)}{a^2}\ \ldots\ldots\ x^{m-(n+1)}\overset{a}{\underset{n+1}{C}}\frac{f(a)}{a^{m-(n+1)}}\right\},$$

or

$$x^{n+1}\overset{a}{\underset{n+1}{C}}f(a)\left\{1 + \frac{x}{a} + \frac{x^2}{a^2}\ \ldots\ldots + \frac{x^{m-(n+1)}}{a^{m-(n+1)}}\right\}$$

$$= x^{n+1}\overset{a}{\underset{n+1}{C}}\left\{f(a)\frac{\dfrac{x^{m-n}}{a^{m-n}} - 1}{\dfrac{x}{a} - 1}\right\}$$

$$= x^{n+1}\overset{a}{\underset{n+1}{C}}\left(\frac{f(a)}{a^{m-(n+1)}}\cdot\frac{x^{m-n} - a^{m-n}}{x - a}\right)$$

$$= x^{n+1}\overset{a}{\underset{m}{C}}\left(f(a)\frac{x^{m-n} - a^{m-n}}{x - a}\right).$$

$$\text{But } \overset{a}{\underset{m}{C}}u = \frac{1}{1.2.\ldots m}\left(\frac{d^m u}{da^m}\right)_{a=0}:$$

consequently the last expression for the remainder is equivalent to

$$\frac{x^{n+1}}{1.2.\ldots m}\cdot\left(\frac{d^m}{da^m}\cdot fa\cdot\frac{x^{m-n}-a^{m-n}}{x-a}\right)_{a=0}.$$

If the development of $f(x)$ is infinite, the terms beyond the $(n+1)^{\text{th}}$ will form the series

$$x^{n+1}\left\{\underset{n+1}{C}f(a) + x\underset{n+1}{C}\frac{f(a)}{a} + x^2\underset{n+1}{C}\frac{f(a)}{a^2} + \text{\&c. to infinity}\right\}.$$

This series is equivalent to

$$x^{n+1}\overset{a}{\underset{n+1}{C}}f(a)\left\{1 + \frac{x}{a} + \frac{x^2}{a^2} + \text{\&c. to infinity}\right\}$$

$$= x^{n+1}\overset{a}{\underset{n+1}{C}}\left\{\frac{f(a)}{1-\frac{x}{a}}\right\} = \frac{x^{n+1}}{1.2.\ldots(n+1)}\cdot\left\{\frac{d^{n+1}}{da^{n+1}}\cdot\frac{f(a)}{1-\frac{x}{a}}\right\}_{a=0}.$$

By means of this expression Maclaurin's theorem becomes

$$f(x) = \{f(a)\}_{a=0} + \frac{x}{1}\left\{\frac{d\,f(a)}{da}\right\}_{a=0} + \frac{x^2}{1.2}\left\{\frac{d^2 f(a)}{da^2}\right\}_{a=0}\ldots.$$

$$+\frac{x^n}{1.2..n}\left\{\frac{d^n f(a)}{da^n}\right\}_{a=0} + \frac{x^{n+1}}{1.2..(n+1)}\left\{\frac{d^{n+1}}{da^{n+1}}\cdot\frac{f(a)}{1-\frac{x}{a}}\right\}_{a=0}\quad ..(1).$$

In like manner we may sum any number of terms of Taylor's series.

For this purpose I observe, that the coefficient of h^n in the development of $f(x+h)$ is

$$\overset{a}{\underset{n}{C}}f(x+a) \text{ or } \overset{a}{\underset{0}{C}}\frac{f(x+a)}{a^n}.$$

Therefore

$$f(x+h) = \overset{a}{\underset{0}{C}}f(x+a) + h\overset{a}{\underset{0}{C}}\frac{f(x+a)}{a} + h^2\overset{a}{\underset{0}{C}}\frac{f(x+a)}{a^2} + \text{\&c.}$$

$$= \overset{a}{\underset{0}{C}}f(x+a)\left(1 + \frac{h}{a} + \frac{h^2}{a^2} + \text{\&c.}\right)$$

Hence, the terms which follow the $(n+1)^{\text{th}}$ are,

$$\overset{a}{\underset{0}{C}}f(x+a)\left(\frac{h^{n+1}}{a^{n+1}} + \frac{h^{n+2}}{a^{n+2}} + \text{\&c.}\right)$$

$$= h^{n+1}\overset{a}{\underset{0}{C}}\frac{f(x+a)}{a^{n+1}}\left(1 + \frac{h}{a} + \frac{h^2}{a^2} + \text{\&c.}\right):$$

$(m+1)$ terms of this last series are equivalent to

$$h^{n+1}\overset{a}{\underset{0}{C}}\left\{\frac{f(x+a)}{a^{n+1}}\,\frac{\dfrac{h^{m+1}}{a^{m+1}}-1}{\dfrac{h}{a}-1}\right\}$$

$$=h^{n+1}\overset{a}{\underset{0}{C}}\left(\frac{f(x+a)}{a^{m+n+1}}.\,\frac{h^{m+1}-a^{m+1}}{h-a}\right);$$

which is equivalent to

$$h^{n+1}\overset{a}{\underset{m+n+1}{C}}\left\{f(x+a)\,\frac{h^{m+1}-a^{m+1}}{h-a}\right\};$$

or, if you will,

$$\frac{h^{n+1}}{1.2.\ldots.(m+n+1)}\left(\frac{d^{m+n+1}}{da^{m+n+1}}\cdot f(x+a)\,\frac{h^{m+1}-a^{m+1}}{h-a}\right)_{a=0}.$$

If the series is infinite, the terms which follow that affected with h^n are,

$$h^{n+1}\overset{a}{\underset{0}{C}}\frac{f(x+a)}{a^{n+1}}\left(1+\frac{h}{a}+\frac{h^2}{a^2}+\&\text{c. to infinity}\right)$$

$$=h^{n+1}\overset{a}{\underset{n+1}{C}}\left\{\frac{f(x+a)}{1-\dfrac{h}{a}}\right\}=\frac{h^{n+1}}{1.2.\ldots.(n+1)}\cdot\left\{\frac{d^{n+1}}{da^{n+1}}.\frac{f(x+a)}{1-\dfrac{h}{a}}\right\}_{a=0}.$$

Hence Taylor's theorem becomes

$$f(x+h)=f(x)+\frac{h}{1}\,\frac{df(x)}{dx}+\frac{h^2}{1.2}\,\frac{d^2f(x)}{dx^2}+\&\text{c.}$$

$$+\frac{h^n}{1.2.\ldots.n}\,\frac{d^nf(x)}{dx^n}+\frac{h^{n+1}}{1.2.\ldots.(n+1)}\left\{\frac{d^{n+1}}{da^{n+1}}\cdot\frac{f(x+a)}{1-\dfrac{h}{a}}\right\}_{a=0}.$$

It is scarcely necessary to observe, that the same method which was employed to sum the Remainder of Maclaurin's series, is applicable to a series which contains negative as well as positive powers.

VI.—SOLUTION OF A PROBLEM IN ANALYTICAL GEOMETRY.

By J. Booth,

Principal of, and Professor of Mathematics in, Bristol College.

The line which joins the points of intersection of two focal right lines, containing a given angle θ, with the conic section, envelopes two conic sections having their foci coincident with the focus of the given section; and if ε and ε' be the eccentricities of the loci, e of the given section, p and p' the parameters of the loci, P that of the given section, we shall have the following relations between the eccentricities and parameters of the three conic sections,

$$\varepsilon^2 + \varepsilon'^2 = e^2, \qquad p^2 + p'^2 = P^2.$$

Let the equation of the given section be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2ex}{a} = \frac{b^2}{a^2} \ldots\ldots\ldots (1),$$

the origin being placed at the focus and the axes parallel to the principal axes of the section.

Let $(y'x')$, $(y''x'')$, be the co-ordinates of the points in which the sides of the given angle θ intersect the curve: the equation of the line passing through those points is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x') \ldots\ldots\ldots (2);$$

or if $\quad y' = mx' \ldots . (3), \quad y'' = m'x'' \ldots . (4),$

be the equations of the sides of the angle, we find, eliminating y', y'', between (2), (3), (4),

$$y - mx' = \frac{mx' - m'x''}{x' - x''}(x - x') \ldots . (5).$$

Let ξ and v denote the reciprocals of the intercepts of the axes of X and Y by the right line (5); then

$$\frac{1}{x'} = mv + \xi \ldots . (6), \qquad \frac{1}{x''} = m'v + \xi \ldots . (7).$$

Now, eliminating (x', y') from the three equations (1), (3), (6), we find the quadratic equation

$$(a^2 - b^4v^2)\, m^2 - 2\,(b^2aev + b^4\xi v)\, m + b^2 - 2aeb^2\xi - b^4\xi^2 = 0 \ldots . (8).$$

Now this is precisely the equation we should have found for m'; hence m and m' are the roots of (8), or

$$m + m' = \frac{b^2aev + b^4\xi v}{a^2 - b^4v^2}, \quad mm' = \frac{b^2 - 2ab^2e\xi - b^4\xi^2}{a^2 - b^4v^2};$$

hence $$m - m' = \frac{2ab\left\{b^2(\xi^2 + v^2) + 2ae\xi - 1\right\}^{\frac{1}{2}}}{a^2 - b^4v^2}.$$

Let the quantity under the radical sign be called M; then

$$\tan\theta = \frac{m - m'}{1 + mm'} = \frac{\pm 2ab\sqrt{(M)}}{a^2 - b^2M};$$

or solving this quadratic equation, we find

$$M = \frac{a^2(1 \pm \cos\theta)^2}{b^2\sin^2\theta},$$

or replacing for M its value, reducing and taking the lower sign, we find

$$\frac{b^4(\xi^2 + v^2)}{\left(b^2 + a^2\tan^2\frac{\theta}{2}\right)} + \frac{2b^2ae\,.\,\xi}{b^2 + a^2\tan^2\frac{\theta}{2}} = 1 \;\ldots\ldots\; (9);$$

had we taken the upper sign, we should have found for the tangential equation of the locus

$$\frac{b^4(\xi^2 + v^2)}{b^2 + a^2\cot^2\frac{\theta}{2}} + \frac{2b^2ae\,.\,\xi}{b^2 + a^2\cot^2\frac{\theta}{2}} = 1 \;\ldots\ldots\; (10).$$

Now, in these equations, as the coefficients of ξ and v are equal, the foci of these sections are at the origin, or coincide with the focus of the given section.

To determine the axes, &c. of these loci. The tangential equation of a conic section whose semiaxes and eccentricity are A, B, and ϵ, the origin of co-ordinates being at a focus and parallel to the axes of the section, is

$$B^2(\xi^2 + v^2) + 2A\epsilon\,.\,\xi = 1 \;\ldots\ldots\ldots\; (a).$$

Comparing this equation (a) with (9), we get

$$B^2 = \frac{b^4}{b^2 + a^2\tan^2\frac{\theta}{2}}, \qquad A\epsilon = \frac{b^2ae}{b^2 + a^2\tan^2\frac{\theta}{2}};$$

hence $\epsilon = e\cos\frac{\theta}{2}$, and $\frac{B^2}{A} = \frac{b^2}{a}\cos\frac{\theta}{2}$, or $p = P\cos\frac{\theta}{2}$.

Had we taken the upper sign, we should have found

$$\epsilon' = e\sin\frac{\theta}{2}, \qquad p' = P\sin\frac{\theta}{2};$$

hence $\epsilon^2 + \epsilon'^2 = e^2$, $\quad p^2 + p'^2 = P^2$:

when θ is a right angle, the two loci coincide.

Had any other point except one of the foci been chosen, we should have found for the locus a curve whose tangential equation would be of the fourth degree; the curve in this particular case separating into two distinct curves, each of which is a conic section.

Had the given section been an equilateral hyperbola, and θ a right angle, the locus would have been found a parabola.

When the given angle θ revolves round the centre instead of the focus, the tangential equation of the locus is

$$\{a^2b^2(\xi^2+v^2)-(a^2+b^2)\}^2 = 4a^2b^2\cot^2\frac{\theta}{2}\{a^2\xi^2+b^2v^2-1\}.$$

VII.—NOTE ON A CLASS OF FACTORIALS.

By D. F. Gregory, M.A. Fellow of Trinity College.

We owe to Vandermonde the interesting Theorem, that Binomial factorials of any order, in which the successive factors differ by a constant quantity, can be expanded in terms of the simple Monomial factorials according to the law of the expansion of Newton's Binomial Theorem. That is to say, that if we put

$$x(x-1)(x-2)\dots(x-n+1) = x^{|n},$$

we shall have

$$(x+y)^{|n} = x^{|n} + nx^{|n-1}y + n\frac{(n-1)}{1.2}x^{|n-2}y^{|2} + \&c.$$

This proposition, which may be proved by various methods, is readily seen to depend on the fact that these factorials are subject to the laws of combination in virtue of which the Theorem of Newton, as applied to ordinary algebraical quantities, is true. And perhaps the Theorem of Vandermonde derives its chief value from its being one of the few examples which we have of the extension of Algebraical Theorems to operations not originally included in the demonstration. The other examples which are known, are the Theorems in the Differential Calculus and the Calculus of Finite differences, which are proved by the method of the separation of the symbols.

Between these however and the Theorem of Vandermonde, there is one marked point of distinction: for whereas in the former Theorems the operation which is subject to the index-operation is different from that which forms the staple of ordinary algebra, while the index-operation is always the same

viz. the operation of repetition, in the latter Theorem the base which is subject to the index-operation is or may be the same as that in ordinary algebra, while the index-operation is different. As any Theorem which will add to examples of this kind must contribute to extend our knowledge of the combination of symbols in the direction in which such an extension seems to be most important, I will offer no apology for occupying a page or two, in demonstrating that a Theorem similar to that of Vandermonde is true of a class of factorials different from that of which he has treated. The factorials to which I allude, are those which are met with in expanding the cosine or the sine of a multiple arc according to the powers of the cosine or sine of the arc itself. These factorials, which are of a somewhat remarkable form, have, like ordinary factorials, an analogy with powers, and the proposition of which I speak is an example of this analogy.

On referring to Vol. II. p. 129 of this Journal, the reader will find the following expressions for cos $n\theta$ and sin $n\theta$ in terms of sin θ when n is an integer,

$$\cos n\theta = \cos n\pi \left\{1 - \frac{n^2}{1.2} v^2 + \frac{n^2(n^2-2^2)}{1.2.3.4} v^4 - \frac{n^2(n^2-2^2)(n^2-4^2)}{1.2.3.4.5.6} v^6 + \&c.\right\},$$

$$\sin n\theta = \cos (n-1)\pi \left\{nv - \frac{n(n^2-1^2)}{1.2.3} v^3 + \frac{n(n^2-1^2)(n^2-3^2)}{1.2.3.4.5} v^5 - \&c.\right\},$$

v being written for sin θ.

Now to exhibit the analogy which the factorials, which are the coefficients of the various terms in these expressions, have with powers, let us represent them by a notation corresponding to that of ordinary factorials, and let us write

$$n = n_{|1},\ n^2 = n_{|2},\ n(n^2 - 1^2) = n_{|3},\ n^2(n^2 - 2^2) = n_{|4},\ \&c.;$$

and generally

$$n_{|r} = n^2(n^2 - 2^2)(n^2 - 4^2)(n^2 - 6^2)..\{n^2 - (r-2)^2\}..r \text{ being even},$$

$$n_{|r} = n.(n^2 - 1^2)(n^2 - 3^2)........\{n^2 - (r-2)^2\}..r \text{ being odd}.$$

This notation being employed, the preceding expressions may be written

$$\cos n\theta = (-)^n \left\{1 - n_{|2} \frac{v^2}{1.2} + n_{|4} \frac{v^4}{1.2.3.4} - n_{|6} \frac{v^6}{1.2.3.4.5.6} + \&c.\right\}..(1),$$

$$\sin n\theta = (-)^{n-1} \left\{n_{|1} v - n_{|3} \frac{v^3}{1.2.3} + n_{|5} \frac{v^5}{1.2.3.4.5} - \&c.\right\}........(2).$$

Now the proposition which we have to demonstrate may be expressed, by means of this notation, in the following manner:

$$(m+n)_{|p} = m_{|p} + p m_{|p-1} n_{|1} + \frac{p(p-1)}{1.2} m_{|p-2} n_{|2} + \&c. + n_{|p}.$$

In the demonstration we must distinguish two cases according as p is even or odd.

1st. Let p be even and $=2r$; then putting $m+n$ instead of n in the series (1) we have

$$\cos(m+n)\theta = (-)^{m+n}\left\{1-(m+n)_{|2}\frac{v^2}{1.2}+\&c.+(-)^r(m+n)_{2r}\frac{v^{2r}}{1.2..r}-\&c.\right\}..(3).$$

But by an ordinary formula of Trigonometry we have

$$\cos(m+n)\,\theta = \cos m\theta \cos n\theta - \sin m\theta \sin n\theta.$$

In this formula, if we substitute for the cosines and sines their equivalent series given in (1) and (2), and if we equate the coefficients of v^{2r}, and then multiply both sides of the equation by $1.2....2r$, we find

$$(m+n)_{|2r} = m_{|2r} + 2r\, m_{|2r-1} n_{|1} + \frac{2r(2r-1)}{1.2}\, m_{|2r-2} n_{|2} + \&c. + n_{|2r},$$

which proves the theorem when p is even.

2nd. Let p be odd and $=2r+1$; then, by means of the series (2) and the formula $\sin(m+n)\theta = \sin m\theta \cos n\theta + \cos m\theta \sin n\theta$, we find on equating the coefficients of v^{2r+1}, and multiplying both sides of the equation by $1.2.3..(2r+1)$,

$$(m+n)_{|2r+1} = m_{|2r+1} + (2r+1)\, m_{|2r} n_{|1} + \frac{(2r+1)\,2r}{1.2}\, m_{|2r-2} n_{|2} + \&c.$$

which proves the theorem when p is odd.

It is easily seen that this result applies equally to factorials of the form

$$n^2(n^2-2^2h^2)(n^2-4^2h^2)....\{n^2-(r-2)^2h^2\},$$

since this last may be written under the form

$$h^r\frac{n^2}{h^2}\left(\frac{n^2}{h^2}-2^2\right)\left(\frac{n^2}{h^2}-4^2\right)....\left\{\frac{n^2}{h^2}-(r-2)^2\right\},$$

which, with the exception of the factor h^r, is the same in form as the factorials which we considered before.

I have not time at present to enter into any further developments of the nature of these factorials, and more particularly into the consideration of their interpretation when the index is negative or fractional: but this is of the less importance, as it is not very likely that expressions of this form will ever be extensively used in analysis; and the demonstration of the preceding theorem is given, not on account of its intrinsic value, but because it illustrates a part of the theory of Algebra which stands most in need of such examples.

VIII.—REMARKS ON THE DISTINCTION BETWEEN ALGEBRAICAL AND FUNCTIONAL EQUATIONS.

By R. L. Ellis, B.A. Fellow of Trinity College.

The distinction which it is usual to make between algebraical and functional equations, will not, I think, bear a strict examination. It is generally said that an algebraical equation determines the value of an unknown quantity, while a functional equation determines the form of an unknown function. But, in reality, the unknown quantity in the former case is a function of the coefficients of the equation, and our object in solving it is simply to ascertain the form of this function. Thus it appears, that in both cases the forms of functions are what we seek.

Let us therefore consider the subject in a more general manner, and endeavour to find a more decided point of distinction. The science of symbols is conversant with operations, and not with quantities; and an equation, of whatever species, may be defined to be a congeries of operations, known and unknown, equated to the symbol zero. Every operation implies the existence of a base, or something on which the operation is performed—in the language of Mr. Murphy, a subject. But the base of an operation is often the result of a preceding one. Thus, in log x^2, the base of the operation log is x^2, itself the result of the operation expressed by the index on the base x. This in its turn may be considered as the result of an operation performed on the symbol unity. But in every kind of equation there is a point at which the farther analysis of symbols into operations on certain bases becomes irrelevant; and thus we are led in every case to recognize the existence of ultimate bases.

To solve an equation of any kind, is to determine the unknown operations by means of the known. If one symbol is said to be a function of another, it is, in reality, the result of an operation performed upon it. Thus the idea of functional dependence pervades the whole science of symbols, and on this idea the following remarks are based.

In order to classify equations, we can make use of two considerations: 1st. The nature of the operations which are combined together; 2nd. The order in which they succeed one another in the congeries of operations which is made equal to zero.

Let us illustrate these remarks by some examples.

If we have an equation of the form

$$x^2 + ax + b = 0 \;\ldots\ldots\ldots\; (1),$$

the bases are a and b; the operations are, first, the unknown one denoted by x, and then certain known ones denoted by the index, the coefficient, &c. All these are what are called algebraical operations.

If again we have an equation of the form

$$\frac{dy}{dx} - x = 0 \,\ldots\ldots\ldots\, (2),$$

the base is x; the operations are, first, the unknown one denoted by y, which is a function of x, then the operation $\frac{d}{dx}$, and lastly, certain algebraical operations. From the presence of the operation $\frac{d}{dx}$, this is called a differential equation. Equations (1) and (2) are discriminated by the nature of the operations combined, on our first principle of classification.

But in one important point these equations agree. In both, the unknown operation is performed immediately on the bases; the known are subsequent to the unknown: but in what are called functional equations this is not so. Thus, in the equation

$$\phi(mx) + x = 0 \,\ldots\ldots\, (3),$$

the base is x, the unknown operation is ϕ, which is performed, not on x, but on the result of a previous operation. In the preceding example the previous operation is known; but this is not essential. Thus in

$$\phi\phi x + x = 0 \,\ldots\ldots\, (4),$$

the previous operation denoted by the right-hand ϕ is unknown. The operation $\frac{d}{dx}$ may enter into equations where the unknown operation is not performed on the base. Thus we may have an equation of the form

$$\phi \frac{d}{dx} \phi x + x = 0 \,\ldots\ldots\, (5).$$

Equations (3), (4), (5), are functional equations; (3), (4), are ordinary functional equations; (5) is a differential functional equation; (3) is said to be of the first order, (4) of the second.

The introduction of the functional notation appears to be sometimes taken as the essence of functional equations; but if we wrote (1) and (2) thus,

$$\{\phi(ab)\}^2 + a\phi(ab) + b = 0 \,\ldots\ldots\, (1)',$$

$$\frac{d}{dx}\phi(x) - x = 0 \,\ldots\ldots\, (2)',$$

they would still be perfectly distinct from (3) or (4) or (5). The name functional equation is not happy; it refers to the notation, and not to the essence of the thing.

A question now arises: To what class shall we refer equations in finite differences? These are generally of the form

$$F(x, y_x, y_{x+1} \ldots.) = 0\ldots\ldots(6),$$

where y_x is an unknown function, say $\phi(x)$ of x; so that (6) may be written thus,

$$F\{x, \phi x, \phi(x+1) \ldots.\} = 0.$$

Here the unknown operation is ϕ, which in the case of $\phi(x+1)$ is performed, not upon the base x, but on $x+1$. Thus it appears, that equations in finite differences are only a case of ordinary functional equations of the first order: and this is the reason why, in researches on functional equations, we perpetually meet with cases in which they may be reduced to equations in finite differences.

The preceding remarks contain, I think, the outline of a natural arrangement of the science of symbols. It is not difficult to overrate the importance of a mere classification; but I hope to be able to show, that the considerations now suggested are not without some degree of utility.

As the distinction between functional and common equations depends on the order of operations, it follows that, when part of the solution of an equation does not vary with the nature of the operation subjected to the resolving process, this part is applicable as much to functional equations as to any other. The special application of this principle to the discussion of a class of differential functional equations will be the object of a subsequent paper.

In the preceding remarks, operations of derivation, such as D, Δ, &c. are supposed to be replaced by functional operations in every case in which this can be effected.

IX.—MATHEMATICAL NOTES.

1. In the Examination Papers for 1834, the following problem is given: "If the chord of a conic section, whose eccentricity is e, subtend at its focus a constant angle 2α, prove that it always touch a conic section having the same focus whose eccentricity is $e \cos \alpha$." A solution of this problem by a peculiar analysis will be found in a preceding article; but the following method may be found not uninteresting.

Let r_1, r_2, be radii vectores to the ends of the chord, $\phi - \alpha$, $\phi + \alpha$, the corresponding angles vectores, p the perpendicular from the focus on the chord;

$$\therefore p \times \text{chord} = r_1 r_2 \sin 2\alpha,$$

$$\therefore \frac{1}{p} = \frac{\sqrt{(r_1^2 + r_2^2 - 2r_1 r_2 \cos 2\alpha)}}{r_1 r_2 \sin 2\alpha} = \operatorname{cosec} 2\alpha \sqrt{\left(\frac{1}{r_1^2} + \frac{1}{r_2^2} - \frac{2}{r_1 r_2} \cos 2\alpha\right)},$$

$$\frac{1}{r_1} = \frac{1 + e \cos(\phi - \alpha)}{l}, \quad \frac{1}{r_2} = \&\text{c.}$$

For $\cos 2\alpha$ put $1 - 2\sin^2 \alpha$; then, by a few obvious steps,

$$\frac{1}{p} = \frac{1}{l \cos \alpha} \sqrt{(1 + 2e \cos \alpha \cos \phi + e^2 \cos^2 \alpha)}.$$

Put $\alpha = 0$; then the chord becomes the tangent, and

$$\frac{1}{p_0} = \frac{1}{l} \sqrt{(1 + 2e \cos \phi + e^2)}.$$

But the general form coincides with this, if we put

$$l \cos \alpha = \lambda \quad \text{and} \quad e \cos \alpha = \epsilon;$$

for then

$$\frac{1}{p} = \frac{1}{\lambda} \sqrt{(1 + 2\epsilon \cos \phi + \epsilon^2)}.$$

Hence p is *generally* the perpendicular on a tangent of an ellipse of eccentricity $e \cos \alpha$. Hence the chord touches such an ellipse. The latus rectum is diminished in the same ratio as the eccentricity.

ε.

2. The relation between the long inequalities of two mutually disturbing planets, may be easily found without having recourse to the development of the disturbing function.

Let m, m', be the masses of the planets, a, a', the major axes of their orbits, n, n', their mean motions, h, h', twice the areas described in 1″; then we have

$$n = \frac{\mu^{\frac{1}{2}}}{a^{\frac{3}{2}}}, \quad n' = \frac{\mu^{\frac{1}{2}}}{a'^{\frac{3}{2}}},$$

μ being the mass of the Sun, in comparison with which the masses of the planets are neglected, so that it is the same for both. Taking the logarithmic differentials of these equations, and replacing the differentials by differences, we find

$$\frac{\Delta n}{n} = -\frac{3}{2}\frac{\Delta a}{a}, \quad \frac{\Delta n'}{n'} = -\frac{3}{2}\frac{\Delta a'}{a'}.$$

But by the principle of the conservation of areas,

$$mh + m'h' = \text{const.}$$

so that $$m\Delta h + m'\Delta h' = 0.$$

Now the orbits being supposed circular, we have

$$h = (\mu a)^{\frac{1}{2}}, \quad h' = (\mu a')^{\frac{1}{2}}:$$

hence $$\frac{\Delta h}{h} = \frac{1}{2}\frac{\Delta a}{a}, \quad \frac{\Delta h'}{h'} = \frac{1}{2}\frac{\Delta a'}{a'}:$$

Therefore we have

$$\frac{\Delta n}{n} : \frac{\Delta n'}{n'} = \frac{\Delta a}{\Delta a'} \cdot \frac{a'}{a} = \frac{\Delta h}{\Delta h'} \cdot \frac{h'}{h} = -\frac{m'}{m}\frac{a'^{\frac{1}{2}}}{a^{\frac{1}{2}}};$$

and $\dfrac{\Delta n}{n}$ and $\dfrac{\Delta n'}{n'}$ are the inequalities due to the disturbances, so that their ratio is thus given.

ε.

3. *Napier's Analogies.*—The form given by Professor Wallace to the mode of solving the spherical triangle whose sides are given, will probably be introduced into all future works on the subject. The corresponding mode of demonstrating Napier's Analogies should not be omitted.

$$M = \sqrt{\{\sin(s-a) \cdot \sin(s-b) \cdot \sin(s-c) \div \sin s\}},$$

$$\tan \tfrac{1}{2}A = \frac{M}{\sin(s-a)}, \quad \tan \tfrac{1}{2}B = \frac{M}{\sin(s-b)}, \quad \tan \tfrac{1}{2}C = \frac{M}{\sin(s-c)},$$

$$\tan \tfrac{1}{2}A \tan \tfrac{1}{2}B = \frac{M^2}{\sin(s-a) \cdot \sin(s-b)} = \frac{\sin(s-c)}{\sin s},$$

$$\frac{\tan \frac{1}{2}A \cdot \tan \frac{1}{2}C \pm \tan \frac{1}{2}B \cdot \tan \frac{1}{2}C}{1 \mp \tan \frac{1}{2}A \cdot \tan \frac{1}{2}B} = \frac{\sin(s-b) \pm \sin(s-a)}{\sin s \mp \sin(s-c)}$$

$$= \frac{\dfrac{\sin}{\cos}\dfrac{c}{2} \cdot \dfrac{\cos}{\sin}\dfrac{a-b}{2}}{\dfrac{\cos}{\sin}\dfrac{a+b}{2} \cdot \dfrac{\sin}{\cos}\dfrac{c}{2}}, \qquad (2s - a - b = c),$$

$$\text{or } \tan\frac{A \pm B}{2} \cdot \tan\frac{C}{2} = \frac{\dfrac{\cos}{\sin}\dfrac{a-b}{2}}{\dfrac{\cos}{\sin}\dfrac{a+b}{2}}.$$

A. D. M.